The concept of number by Karl Weierstraß and its first publication, viz. within a school program by Ernst Kossak

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Abstract

Karl Weierstraß (1815–1897) presented his concept of real number and its generalizations in his lectures at Berlin university, but did not publish it in printed form. Instead of this, other mathematicians wrote publications on his theory, starting off with Ernst Kossak (1839–1892) in 1872, who was a teacher at the Friedrichswerdersche Gymnasium in Berlin then. This treatise appeared within the yearly program of Kossak's school. By choosing this way of publication, he addressed rather teachers at other secondary schools or even a generally interested public than mathematicians at universities. In fact, academic circles reacted unfriendly: Hermann Amandus Schwarz (1843–1921) informed Weierstraß about flaws in it. Later on, Weierstraß himself expressed his negative views on it to Gösta Mittag-Leffler (1846–1927). Whereas these criticisms were only written in private letters, Gottlob Frege (1848–1925) directly attacked Kossak in his 1884 book on the foundations of arithmetic. Frege's criticism, however, directly referred to the ideas and methods of Weierstraß proper, whereas the unfriendly remarks of Weierstraß and, in particular, Schwarz may root in a quarrel on an academic position.

Keywords: concept of number, Karl Weierstraß, real numbers, complex numbers, hypercomplex numbers, Ernst Kossak, school program, Hermann Amandus Schwarz, Gottlob Frege

Introduction

Parallel to its title, the contents of this contribution are divided into three sections: "The concept of number by Karl Weierstraß" – "and its first publication, viz. within a school program" – "by Ernst Kossak":

The concept of real number was formalized by several authors, like Georg Cantor (1845–1918), Richard Dedekind (1831–1916), Eduard Heine (1821–1881), and Charles Méray (1835–1911) in the years around 1870. But, for example, even if Dedekind had developed his concept of "cuts" while preparing a lecture course on mathematics for engineers, there is no evidence that he ever used them in his academic teaching. Contrary to this, Karl Weierstraß (1815–1897) presented his ideas on the concept of number, which he had developed in the 1860ies, every two years in his well-attended introductory lecture course on complex analysis at Berlin university. Starting off from a concept of positive integer number referring to everyday life, he constructed not only the real but also the complex and even hypercomplex numbers. This laid the fundaments for his other lecture courses, on elliptic functions, on Abelian functions, or on the calculus of variations, for

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example. However, Weierstraß did not publish a textbook on the complete course or even parts of it, but presented his ideas only orally. Surely, one can find editions of lecture notes of his higher courses in his *Mathematische Werke* (Weierstraß 1894–1927). But, contrary to this, his introductory lecture series with his theory of numbers had to wait until the 1980ies for publication, e. g., (Weierstraß, 1988a), (Weierstraß, 1988b).

These circumstances naturally harmed the dissemination of his concept of number outside Berlin which may explain that one cannot find his definition of real number in current analysis texts explicitly. However, from the mathematical standpoint, one can interpret his central idea as a straightforward generalization of the representation of real numbers by decimal fractions, namely that, instead of merely $1/10$, $1/100$, $1/1000$, $\ldots$, arbitrary unit fractions were admitted for the representation.

Already during the lifetime of Weierstraß other mathematicians reacted on his reluctance to the printing press by publishing texts on his concept of number which were based on notes taken from his lectures. The first of them was Ernst Kossak (1839–1892) in 1872, who had attended the introductory lecture course of Weierstraß in the winter semester 1865/66 and was a teacher at the Friedrichswerdersche Gymnasium at Berlin then. Kossak's text *Die Elemente der Arithmetik* (Kossak, 1872b) did not appear in a scientific journal or as an independent printed work but within of a program of his school ("Schulprogramm"). So one can consider this publication as an example of the transportation of knowledge from university to secondary schools.

However, if one looks into the literature, both of philosophy of mathematics and of its history, the reputation of Kossak's text shows some stains: Gottlob Frege (1848–1925) attacked Kossak by name in his *Die Grundlagen der Arithmetik* (Frege, 1884, §§ 63, 102, 103) because of the definition of equality of numbers and the use of additional units besides the counting unit, e. g., the imaginary one. Furthermore, Gösta Mittag-Leffler (1846–1927) even published in (Mittag-Leffler, 1920, p. 206) a letter written by Weierstraß on April 5, 1885 in which the latter complained that Kossak had "messed up" ("verhunzt") his theory. Last, but not least, Pierre Dugac brought to light a letter of Hermann Amandus Schwarz (1843–1921) to Weierstraß in which he had given a pages long list of errors in Kossak's treatise only a few weeks after its publication (Dugac, 1973, pp. 143–144). Therefore, one may get the impression that Kossak only gave a faulty, not reliable exposition of the ideas of Weierstraß, in particular his definition of real numbers. But a closer look mitigates this verdict:

On the one hand, the ideas that Frege criticized had been taken over by Kossak directly from Weierstraß. On the other hand, the relation of Schwarz and Weierstraß to Kossak as a person turns out to be of interest: Kossak's name can be found in lists containing students of Weierstraß like (Behnke, Kopfermann, 1966, p. 37) and (Bölling, 1994, p. 41) so that one might think that Kossak was such a student. However, his vita reveals that this was not the case: He took the overwhelming part of his academic studies at Königsberg university, mainly with

Friedrich Richelot (1808–1875), but also influenced by at least the writings of Georg Rosenhain (1816–1887). Kossak had also taken his teacher exam for Gymnasia at Königsberg and moved to Berlin only for his preparatory year and his following position as a school teacher. There is no evidence that he visited any other lecture course of Weierstraß than the one that he used as basis for his treatise (Kossak, 1872b). Finally, Kossak took a doctorate, but not at Berlin university under the supervision of Weierstraß, but at Göttingen university *in absentia* with his Ph. D. thesis assessed by Alfred Clebsch (1833–1872).

Therefore, Kossak was rather a student of what was named the "Königsberg school" of mathematics at that time. Perhaps his greatest fault in the eyes of Schwarz and Weierstraß was that this person with scientific roots not in Berlin was on his way to gain a professorship for mathematics at the Gewerbeakademie in Berlin, a position of the kind that Weierstraß had held some 15 years before. This makes Kossak another example of a teacher at a secondary school who became professor, in this case at a technical institute in Prussia. Seen from this point of view, however, Kossak's publication (Kossak, 1872b) of the concept of number by Weierstraß becomes just one step in a career path.

1. "The concept of number by Karl Weierstraß"

This notion covers a rather extended area in the whole: Weierstraß started off with the positive integer numbers and constructed from them the positive rational numbers. The next step was his construction of the positive real numbers, analogously the negative ones and then the real numbers as such. (This part of the theory has intensively been studied both by his contemporaries and historians of mathematics, cf., e. g., (Gericke 1970, Subsect. 5.5.1), (Dugac 1973, Sect. 4), and (Spalt 2023).) However, Weierstraß did not end up with this step but also explained how to get the imaginary numbers in order to arrive at the usual complex numbers which he needed as basis of his theory of analytic functions. Furthermore, he even advanced further on to "hypercomplex" numbers, in modern view: finite dimensional algebras over the real numbers.

In greater detail, Weierstraß proceeded as follows:

1.1 Numbers, complex numbers and "Aggregate"

Weierstraß starts off with the notion of natural numbers in a rather propaedeutic way, without any axiomatization (Weierstraß, 1988a, p. 6):

> "Die arithmetischen Grundoperationen beziehen sich zunächst auf <u>ganze positive Zahlen</u>. Zu diesen gelangen wir durch die Betrachtung einer Vereinigung gleichartiger Dinge."
>
> (= "The basic arithmetic operations initially relate to <u>integer positive numbers</u>. We arrive at these by considering a union of similar things.")

Such unions of "similar" things are generated by one single unit ("Einheit"), usually identified with 1. But Weierstraß also allows that "dissimilar" ("ungleichartige") things are put together (Weierstraß, 1988a, p. 7):

> "Es seien z. B. in einem Haufen Menschen Männer, Weiber und Kinder enthalten, die wir getrennt zählen wollen. Wir stellen dann das Vorkommen jeder einzelnen Gattung durch eine Zahl dar und fügen einer jeden Zahl ein Zeichen zur Charakterisierung der Einheit, aus der sie gebildet ist, bei. Auf diese Weise entsteht z. B. eine Zahl
> $$12\,\alpha + 37\,\beta + 2\,\gamma\,.$$
> Eine solche aus mehreren verschiedenen Einheiten gebildete Zahl ist ein Complex von Zahlen und heisst daher eine complexe Zahl."
> (= "For example, a crowd of people includes men, women, and children, whom we want to count separately. We then represent the occurrence of each genus by a number, and add to each number a character to characterize the unit of which it is composed. In this way a number like
> $$12\,\alpha + 37\,\beta + 2\,\gamma$$
> is generated. Such a number formed from several different units is a complex of numbers and is therefore called a complex number.")

In another lecture series he gives a somewhat less problematic example from everyday life, viz., coins like "a Reichsgulden b Groschen c Pfennige", cf. (Weierstraß 1988b, p. 3). (For the history of the notion of "complex number" cf. (Ferreira, Schubring, 2022, Sect. 4).)

Weierstraß calls these unions of copies of the same or of different units "Aggregate" (= "aggregates"). Note that, even if he calls their constituents "Elemente" (= "elements"), these aggregates are not sets since they can contain several copies of the same unit. (In modern terms: He considers multisets.)

1.2 The construction of the rational, the real, and the (usual) complex numbers

This concept becomes powerful when one puts copies of different but algebraically related units together into the same aggregate. As a first, purely algebraic step, Weierstraß defines the positive rational numbers by taking a main unit ("Haupteinheit") u and, for each natural number n an auxiliary unit $\frac{1}{n}u$ with the property that the n-fold multiple of $\frac{1}{n}u$ equals u so that it defines the nth exact part of u ("n-ter genauer Theil") Using this relation, the rules for building such fractional parts of u for different denominators such as $\frac{1}{m}\left(\frac{1}{n}u\right) = \frac{1}{m \cdot n}u$ are deduced from the corresponding rules for the positive integer multiples of u under the explicit assumption that the laws of arithmetic also hold for the exact parts The positive rational multiples of u can then be defined by the aggregates which contain altogether only finitely many copies of the main unit u and its nth exact parts $\frac{1}{n}u$ where also different values of n are admitted, as, for example, $n = 2$, $n = 3$, $n = 4$, ….

However, an aggregate can also contain infinitely many elements. Assuming at first $u = 1$, a given aggregate defines a positive real number if there is a constant M with the property that for each finite collection of elements taken from this aggregate the sum of its elements – which can be executed by finitely many arithmetic operations – is below this M. This definition obviously fits well to the theorem, often associated with the names of Bernard Bolzano (1781–1848) and Weierstraß, that each monotonically increasing sequence of (positive) real numbers which is bounded from above has a limit. The construction can easily be generalized to arbitrary units u instead of $u = 1$ in order to define positive real multiples of u. Working with this definition in formal details like the relations "equal to" or "greater than" is, on the whole, comparably as technical as for the other constructions of the real numbers.

Up to now, all constructions restrict to positive multiples of u. In order to handle also other types of numbers, Weierstraß turns to algebraic methods again: For the negative numbers, he chooses an opposite unit u' ("entgegengesetzte Einheit") for the main unit u, which annihilates u when added:

$$u + u' = 0 = u' + u\,.$$

Then he considers aggregates that consist of copies u and u' together with their respective exact fractional parts. Such an aggregate defines a real multiple of u if and only if both the copies of u and its exact parts together define a positive real multiple of u and if the analogous property holds for the copies of u' and its exact parts together. For the special case $u = 1$ (and $u' = -1$, hence) this gives his definition of real numbers, also called his theory of irrational numbers ("Theorie der irrationalen Zahlen").

The complex numbers can then be defined by a simple generalization of this procedure: Instead of considering aggregates only made of copies of $u = 1$ and $u' = -1$ one choses a further unit, i, with the property that $i \cdot i = -1$ and also an opposite unit $i' = -i$ for it. Then an aggregate formed out of copies of 1, -1, i, and $-i$ and their respective exact parts gives rise to a (usual) complex number if it fulfills the boundedness condition for each of the four units and its exact parts separately.

1.3 Hypercomplex numbers and uniqueness questions

This construction is not restricted to the main units 1 and i but can be transferred to any finite number of main units, which leads to hypercomplex number systems. Weierstraß requires that an addition and a multiplication are defined on them which are linear over the real numbers, commutative and distributive, and fulfill the laws $a + (b + c) = b + (a + c)$ and $a \cdot (b \cdot c) = b \cdot (a \cdot c)$ (instead of the law of associativity). Furthermore, he discusses the possibility of dividing one element by another. Such general number systems had been under discussion at least since the remark of Carl Friedrich Gauß (1777–1855) that he could show (Gauss 1831, p. 178),

> „warum die Relationen zwischen Dingen, die eine Mannigfaltigkeit von mehr als zwei Dimensionen darbieten, nicht noch andere in der allgemeinen Arithmetik zulässige Arten von Grössen liefern können"
> (= "why the relations between things, which present a manifold of more than two dimensions, cannot yield other kinds of quantities which are admissible in general arithmetic").

Weierstraß studied the properties of these number systems in his lectures from the 1860ies onwards in an increasingly detailed way, culminating in the Dedekind-Weierstraß theorem on the structure of (in modern terminology) finite-dimensional associative and commutative real algebras without nilpotent elements. (For details and also his quarrel with Dedekind concerning this result cf. (Ullrich, 2017, esp. Subsect. 4.3).) But already in his lectures course in the winter semester 1865/66, which Kossak attended, Weierstraß stated and proved that each two-dimensional ring extension of the real numbers without zero divisors is equivalent to the field of complex numbers, i.e., is isomorphic to it as a real algebra.

2. "its first publication, viz. within a school program"

2.1 School programs and Kossak's publication

Later expositions of the concept of number by Weierstraß were published as articles in scientific journals like the one by Salvatore Pincherle (1853–1936) (Pincherle, 1880), within a book like the one by Otto Biermann (1858–1909) (Biermann, 1887) or even as a separated book like the one by Victor von Dantscher (1847–1921) (von Dantscher, 1908). On the contrary, Kossak published his treatise (Kossak, 1872) within a school program ("Schulprogramm"), in his case of the Friedrichswerdersche Gymnasium at Berlin for the school year from 1871 to 1872, where he worked as teacher.

These school programs were specific for the secondary school system in several parts of Germany, in particular Prussia: They were printed booklets which were published annually by a secondary school, in particular, a Gymnasium, and listed the important events at that school during the last year, like final examinations or the entry and exit of teachers to the college. However, more important for the present considerations was that by a decree of the Prussian Ministry of Culture from the year 1824 it was obligatory that the school program contained one scientific article, called program treatise ("Programmabhandlung"), which had to be written by one of the teachers of that school. The idea was that over the years each teacher would write such a treatise in his own discipline. The Humboldtian reform of the educational system was based on the concept that the teachers at a Gymnasium were scholars, and by these treatises they should prove this to their colleagues at their school and also at other schools with which the programs were exchanged. Furthermore, it was a way to present the school to the local educated public, for example to the parents of the school (at that time only) boys. (For a more detailed exposition on school programs and program treatises in mathematics and the natural sciences see (Schubring, 1986, in particular Part I) and (Schubring, 2021).)

Therefore, by publishing his treatise (Kossak, 1872b) in a school program Kossak addressed less mathematicians at universities but rather his colleagues as school teachers or even a generally interested public. This assumption is supported by the fact that almost half of the treatise is devoted to a historical overview about the development of arithmetic (Kossak, 1872b, § 1) which Kossak added to the presentation of the ideas of Weierstraß.

This presentation proper takes place in (Kossak, 1872b, § 2): Kossak starts off with one unit and its exact parts. The next point is his discussion of irrational numbers where he rather refers to authors from Greek antiquity than to Weierstraß himself and the details of his technical considerations. Kossak, however, clearly stresses the importance of unconditional convergence of series of positive summands (Kossak, 1872b, p. 21). Only then he introduces opposite units. Following this, about after one half of the presentation of the ideas of Weierstraß, Kossak defines hypercomplex numbers and their general properties, but quickly turns to the case of only two main units, which leads him to the usual complex numbers. Furthermore, he shows that the case of three main units does never lead to, in modern terminology, a field.

2.2 Attention paid to program treatises of Weierstraß

By a look at bibliographic databases one can find that a considerable number of copies of school programs have still remained. But it is difficult to find out to which degree they were studied and how they were valued at the time of their publication. This can exemplarily be illustrated by means of the two protagonists of the present text, Weierstraß and Kossak:

Weierstraß had worked as secondary school teacher from 1842 to 1848 at the Progymnasium – which only comprised of the lower classes of a complete Gymnasium – in Deutsch Crone (West Prussia), today: Wałcz (Poland), and from 1848 to 1855 at the Gymnasium in Braunsberg (East Prussia), today: Braniewo (Poland). During these 13 years he published not less than three treatises in the programs of his respective schools: For the first treatise, (Weierstraß, 1843), August Leopold Crelle (1780–1855) had even offered him his *Journal für die reine und angewandte Mathematik* for publication. But, for what reasons whatsoever, Weierstraß chose the program of his school instead. (See (Ullrich, 2007) for details of this story.) The next one, (Weierstraß, 1845), was published for the simple reason that the colleague who had been in charge of the program treatise had not been able to finish his text so that Weierstraß substituted it by a text on pedagogy which he had written for his teacher exam at Münster in 1841.

Considerably more important was the third treatise, *Beitrag zur Theorie der Abel'schen Integrale* (Weierstraß, 1849) in which Weierstraß sketched his ideas to solve the Jacobi inversion problem for hyperelliptic functions, a problem that had occupied him since his student days and the solution of which would launch his academic career. However, for this it would take a further publication, namely (Weierstraß, 1854), since in this case there is a written proof of the lack of attention paid to a program treatise by university mathematicians: In June 1855 Gustav Lejeune Dirichlet (1805–1859) was going to move away from Berlin university to Göttingen and wrote a letter with suggestions for his successorship which he had worked out together with a committee: Ernst Eduard Kummer (1810–1892) was placed first and Friedrich Richelot (1808–1875) second. Then the committee named three further mathematicians that were considered highly suitable for a full professorship because of their scientific achievements, among them Weierstraß. On him, Dirichlet started his report with the following words (Biermann, 1988, p. 270):

„H. *Weierstraß*, dessen seltenes Talent die Mathematiker schon vor 6 Jahren in einem vortrefflichen Schulprogramm hätten erkennen können, wenn diese Schrift nicht leider ganz unbekannt geblieben wäre, […].“

(= “Mr. *Weierstraß*, whose rare talent the mathematicians could have recognized already 6 years ago in an excellent school program, if this writing had not unfortunately remained completely unknown, […].”)

2.3 Attention paid to the program treatise of Kossak

Contrary to the case of Weierstraß, Kossak’s program treatise (Kossak, 1872) was noticed within the academic community, but not in a way that will have suited him.

2.3.1 The criticism by Schwarz

The first person to react was Hermann Amandus Schwarz, the favorite student of Weierstraß, who had moved to a full professorship at the Eidgenössische Polytechnikum (today: ETH) in Zürich in 1869. On June 20, 1872, only few weeks after the publication of (Kossak, 1872), he wrote a letter to Weierstraß. Since the text of this letter in German has already been published in (Dugac, 1973, pp.143–144), we will only reproduce an English translation of it. (Note that the handwritten version of this letter is not contained in the file No. 1254 of the Schwarz estate in the Archiv der Berlin-Brandenburgischen Akademie, but in the file No. 1320, a book with copies of letters written by Schwarz.):

“Hottingen near Zurich, June 20, 1872

Dear Professor!

Since sending my last letter to you [on May 18, 1872], I have become acquainted with Mr. *E. Kossak*’s program treatise entitled ‘Die Elemente der Arithmetik’, in the preface to which reference is made to your lectures and your permission for publication. Permit me to write a few words about this treatise, at the risk of not telling you anything new with them.

First of all, I say that I am very sorry to see your lecture so distorted in this program. I would have expected better from Mr. *K.*, whom I think I know personally, but I cannot explain how someone who has only a reasonably good mathematical education can make a blunder like the one made on p. 25 of the program that Mr. *K.* has got into debt. Right at the beginning I miss the establishment of the point of view by which the limitation of the area of numbers is determined, the requirement of unlimited reversibility of the arithmetic operations generally introduced by addition and multiplication, I also miss the emphasis on the requirement that all formal laws valid for arithmetic with integer numbers should be maintained. Mr. *K.* seems to have no knowledge of the conditions that must exist between the constants of the law of multiplication and which result from the requirement $a \cdot (b \cdot c) = b \cdot (a \cdot c)$, at least he mentions none of it at the position where this should have been mentioned. The fact that there are numbers $e_0 = \frac{a}{a} = \frac{b}{b}$ that play the role of the absolute unit for every law of multiplication that allows division in general, seems unknown to the work. Mr. *K.*'s insufficient acquaintance with the subject he is dealing with is shown most clearly by the argumentation on pag. 25. From the fact that a quadratic form of two variables can only be equal to zero if both variables individually have the value zero, it can never be concluded that it does not contain the product of the two variables. Therefore, the whole investigation on the numbers depending on two basic elements is imprecisely and rather carelessly reported. Nothing new could be found in this area after your lecture. The task that the author had to solve consisted exclusively of a careful and correct presentation of the thoughts you published in that lecture; in this respect, *K.*'s presentation does not seem to me to be able to meet even moderate requirements. The historical remarks from *Humboldt*'s treatise and from the arithmetic books of Euclid one would have liked to have forgiven the author."

If one looks into (Kossak, 1872) one finds that in fact Kossak is a little bit too vague with his formulations: He does not explicitly state that addition and multiplication have to be commutative or the law $a \cdot (b \cdot c) = b \cdot (a \cdot c)$. But he writes, in highlighted font, (Kossak, 1872, p. 23):

„Wenn complexe Zahlen in die Arithmetik eingeführt werden sollen, so lässt man die Grundelemente derselben zunächst unbestimmt, und stellt die Frage, welche Grundeinheiten werden in der Arithmetik unter der Bedingung zulässig sein, dass die Rechnungsoperationen, welche für ganze (positive) Zahlen aufgestellt sind, auch für complexe Zahlen mit diesen Grundeinheiten aufrecht erhalten werden können".

(= "If complex numbers are to be introduced into arithmetic, the basic elements of the same are initially left undetermined, and the question is asked which basic units will be permissible in arithmetic under the condition that the arithmetic operations which are set up for integer (positive) numbers can also be maintained for complex numbers with these basic units".)

Furthermore, Kossak mentions that opposite units are introduced in order to make subtraction always possible (Kossak, 1872, pp. 22, 23–24). In fact, the argument on "pag. 25" that Schwarz refers to has a gap, but this can be mended by changing the units under consideration by means of a linear transformation.

Of course, one can argue that in a text for a general public the mathematical exposition should be stainless, in particular when the author does not present new results found by himself, but reproduces findings of another person. But when reading the criticism by Schwarz as a whole one gets the impression that his central accusation towards Kossak is that the latter tries to (mis)use the name of Weierstraß in order to upgrade his publication. At any rate, Schwarz does not criticize Kossak's exposition of the theory of real numbers, even if – at least from the modern viewpoint – one may find that it is written rather scarcely, in particular with respect to the ideas of Weierstraß (Kossak, 1872, pp. 19–21).

2.3.2 The criticism by Weierstraß

The correspondence between Schwarz and Weierstraß contains no indication of a direct reaction of the latter. However, in a letter to Mittag-Leffler on April 5, 1885 he wrote (Bölling, 2016, p. 73):

> "Sie wissen, wie meine Einleitung in die Functionentheorie von den Herren Kossak [in the original: Kossack] und Pincherle verhunzt worden ist."
> (= "You know how my introduction to the theory of functions was messed up by Messrs. Kossak and Pincherle.")

This quote is given in Mittag-Leffler's article (Mittag-Leffler, 1920, p. 206) under omission of Pincherle's name, presumably on purpose, and with a wrong date of the letter as Reinhard Bölling has pointed out and corrected (Bölling, 2016, p. 73, fn. 50).

However, in order to evaluate this verdict, one should realize that Weierstraß was not happy with any exposition of his theory, even with his own one, cf., e. g. (Bölling, 2016, pp. 73–74), where one can also find his criticism of (Biermann, 1887).

2.3.3 The criticism by Frege

The harsh remarks on Kossak's treatise (Kossak, 1872) quoted above were written up in letters and therefore not meant for the general public. But already in 1884 Gottlob Frege attacked Kossak by name in his *Die Grundlagen der Arithmetik* (Frege, 1884). In detail, he criticized that Kossak had defined the equality of two integer numbers by means of "eindeutige Zuordnung", i. e., bijections (Frege 1884, § 63, p. 74, footnote *), that he had postulated the general possibility of subtractions (Frege 1884, § 102, p. 111, footnote *) and that he carelessly dealt with an imaginary unit (Frege 1884, § 103, pp. 112–113, footnote *).

Deliberately or not, at this occasion Frege hit Kossak, but meant Weierstraß: All these ideas had been presented by the latter in his lectures. Only in the second volume of his *Grundgesetze der Arithmetik* (Frege, 1903), Frege used also other sources like (Biermann, 1887) and sets of lecture notes for the concept of number by Weierstraß and also made clear that he tried to analyze the latter's ideas. (As an aside: The introduction of von Dantscher's exposition of the concept of number by Weierstraß (Von Dantscher, 1908) contains replies to Frege's criticism in (Frege, 1884) and (Frege, 1903).)

3. "by Ernst Kossak"

The best source on Kossak's life published up to now seems to be the obituary (Lampe, 1903) by Emil Lampe (1840–1918). Even sources that are reliable in general like (Bölling, 1994, p. 41), also (Behnke, Kopfermann, 1966, p. 37), have to be taken with care because of some incorrect pieces of information on him. ((Biermann, 1988) does not mention him at all, which is an eminent indication that he was not really associated to the Berlin university.) A presentation of Kossak's complete life will be published online in due time (Ullrich, 2024). Therefore, in the sequel only information will be given that pertains to his treatise (Kossak, 1872) in the wider sense.

3.1 Youth and university studies

Ernst (August Martin) Kossak was born on January 16, 1839 in Friedland (East Prussia), today: Prawdinsk (Oblast Kaliningrad, Russia), about 40 km away from Königsberg, today: Kaliningrad. Before finalizing the Gymnasium he left it in 1857 in order to study mechanical engineering, but returned to it in 1859 in order to take his final school exam ("Reifeprüfung"). In the same year he started his studies in mathematics and physics at Königsberg university, where he was mainly influenced by the two full professors Richelot in mathematics and Franz Neumann (1798–1895) in physics. Furthermore, also the associate professor in mathematics Rosenhain taught at this place, who had been a serious competitor to Weierstraß in the area of hyperelliptic functions in the 1850ies. (In Dirichlet's report for his successorship mentioned before, Rosenhain was named before Weierstraß.) But, perhaps because of the political problems that he had in the course of the 1848/49 revolution, Rosenhain had meanwhile seized to be a brilliant scientist. However, his paper on ultraelliptic functions of the first kind (Rosenhain, 1851) that had won the 1846 prize of the Paris academy was quoted in several of Kossak's writings.

For example, Kossak's written thesis for the teacher exam for Gymnasia ("Staatsexamen") was based on results on the addition theorem for ultraelliptic functions from (Rosenhain, 1851). It was highly praised by Richelot in a written assessment, but Kossak just finished his teacher exam at Königsberg on April 1, 1865 and received no doctorate there.

3.2 Teacher and Privatdozent at Berlin

Immediately afterwards Kossak moved to Berlin where he started his preparatory year as teacher at the Dorothenstädtische Realschule in summer 1865 and, since October 1865, worked as a teacher at the Friedrichswerdersche Gymnasium until the beginning of 1872. Furthermore, he attended the lecture course "Principien der Theorie der analytischen Functionen" of Weierstraß introducing into the theory of analytic functions at Berlin university during the winter semester 1865/66, where he took the notes on which the treatise (Kossak, 1872b) is based. However, he did not pay the student fees for it. (I am grateful to Gert Schubring for having me communicated this result of his research.) Therefore, one gets the impression that he did not intend to get into close contact with Weierstraß.

Instead, from October 1, 1868 to September 30, 1869 Kossak had a part time position as a teacher for physics at the military academy in Berlin. And in 1869 he received his Habilitation, the exam for teachers in the tertiary system, for mathematics at the Gewerbeakademie in Berlin, became private lecturer ("Privatdozent") and gave lectures on part time basis ("Lehraufträge") there. Before its renaming in 1866 the Gewerbeakademie had been called "Gewerbeinstitut". At this institution Weierstraß had earned his living as a professor from 1856 until 1861/64. In 1879 it would become one of the founding blocks of the "Königlich Technische Hochschule zu Berlin in Charlottenburg" (TH Charlottenburg), which today is the Technical University Berlin.

Note that in 1869 Kossak had not (yet) received a Ph. D., which usually was a necessary condition for a Habilitation at a university. The Gewerbeakademie had received the right to grant this academic degree and by this generate private lecturers as junior staff for its teaching positions in 1866. But it did not establish formal regulations for the bestowal of the degree. Even after the formation of the TH Charlottenburg in 1879 it took five further years until such regulations were issued. In particular, before 1884 there was no duty that a candidate had received a doctorate before passing the Habilitation at the Gewerbeakademie. (Remarkably, it took until 1899 that the TH Charlottenburg received the right to grant doctoral degrees for engineers and even until 1924 that it received this right for mathematics and the natural sciences).

3.3 Doctorate and publications

But even after his Habilitation Kossak must have gotten the impression that a doctoral degree would be adequate: Maybe the institutions or his colleagues indicated to him that this was apt for a private lecturer, maybe he felt that such a degree was necessary to get a full time instead of a part time position at the Gewerbeakademie. (The future would show that this was a good guess.)

Somewhat surprising, however, he did receive his doctorate not under Weierstraß, not even in Berlin, but in Göttingen, obviously without ever having studied there. On May 31, 1871 Kossak submitted his request for a doctorate to the Philosophische Fakultät at Göttingen augmented with the wish to receive the degree *in absentia*, i.e., without an oral exam, since he could not leave Berlin because of his position as a teacher at the Friedrichswerdersche Gymnasium and as a private lecturer at the Gewerbeakademie (UniA GOE Phil. Fak. 156, Vorgang Promotion Ernst August Kossak). Obviously, this request was no surprise for the relevant persons at Göttingen, because already on June 4, 1871 is was noted in the files that the submitted treatise had been examined by Clebsch (ibid.).

Clebsch had also been Richelot's student at Königsberg, but distinctly before Kossak's time there. Another connection from Kossak to Göttingen was the fact that Rosenhain was corresponding member of the Göttingen "Wissenschaftliche Gesellschaft" since 1856. That Kossak had to take some preparations, so to speak, for his doctorate becomes plausible when one looks at the submitted treatise *Das Additionstheorem der ultra-elliptischen Functionen erster Ordnung* (Kossak, 1871): It deals with the same results as his written thesis for the teacher exam (which had been praised by Richelot and not been published at that time) and gives only new methods of proof for them. After all, Kossak received his doctorate in mathematics on August 31, 1871 *in absentia* at the university in Göttingen. (Note that Weierstraß will have heard about the circumstances of this process and may have used it as a blueprint when only few years later his student Sofia Kovalevskaya (1850–1891) should receive a doctorate.)

In the years 1871 and 1872 Kossak did not only receive a doctorate but also published several treatises: his dissertation (Kossak, 1871), of course, then the treatise on the concept of number by Weierstraß (Kossak, 1872b), and, finally, a contribution *Zur Theorie der elliptischen Transzendenten* (Kossak, 1872a), which, alike (Kossak, 1871), is based on ideas from (Rosenhain, 1851). These are the only publications by Kossak of which the "Jahrbuch über die Fortschritte der Mathematik" knows, even if it was founded in 1868!

3.4 Professor for higher mathematics at the Gewerbeakademie

The reason for this outburst of academic activities by Kossak might have been a professional perspective: From its foundation in 1821 onwards, teaching in mathematics had been established at the Gewerbeinstitut, in particular for descriptive geometry, even if on part-time positions. Only in 1856, a professorship for higher mathematics, in particular analysis, was created, more or less personally for Weierstraß. On this position Siegfried Aronhold (1819–1884) substituted Weierstraß from the latter's breakdown in the winter term 1861/62 onwards, became his successor in 1864 and remained on this position until his retirement in 1883. (See (Knobloch, 1998) for the general history of the mathematical positions at the Gewerbeinstitut and the Bauakademie.)

Aronhold's colleague at the Gewerbeakademie in the years around 1870 was Elwin Bruno Christoffel (1829–1900) who had received his Ph. D. at the university in Berlin in 1856 under Ernst Eduard Kummer (1810–1893) and had also taken his Habilitation there in 1859. In 1862 he followed a call to a professorship at the Eidgenössische Polytechnikum in Zürich but returned to Berlin in 1869 in order to fill the second professorship for higher mathematics which had been created at the Gewerbeakademie on the initiative of Franz Reuleaux (1829–1905), its director since 1868. However, Christoffel did not feel too comfortable at Berlin, presumably since he did not additionally get a position as extraordinary professor at the university. Therefore, he left Berlin again, following a call to the newly founded German Reichsuniversität at (then) Straßburg, today again: Strasbourg, on April 1, 1872.

For Kossak who was working together with Christoffel, even if on a part time position, this development may not have been a surprise. Therefore, his activities in order to strengthen his academic standing in the years 1871 and 1872 could have been motivated by the idea to become Christoffel's successor at the Gewerbeakademie. But there was a further person in the play who was probably also early aware of the necessity to find a successor to Christoffel: Schwarz had been the successor to the latter at the Eidgenössische Polytechnikum in Zürich, and, even if he was eleven years younger than him, he seemed to have felt on equal terms with him, maybe because of their equal academic qualifications. In particular, Schwarz had negotiated with the Eidgenössische Polytechnikum as long as he got the same payment as Christoffel. Maybe, he himself was interested in the position at the Gewerbeakademie. (Seen from hindsight, this might be seen as exaggerated because he would be full professor as indirect successor to Gauß, Dirichlet, and Riemann only very few years later. But at the time under consideration no one could have expected that Clebsch, who presently held this position and who was not even forty years old, would die on November 7, 1872 of diphtheria.) Maybe, Schwarz also thought that another student of Weierstraß should get the position. In any case, this may have incited his criticism of the writing (Kossak, 1872b) of a person whom he will have seen as trying to bask in the glory of Weierstraß even he was not his student.

At any rate, things proceeded not totally smooth for Kossak: Surely, on January 15, 1873 he got a full time teaching position at the Gewerbeakademie which was related to the teaching duties covered by Christoffel before and which Kossak filled from April 1, 1873 onwards. But only on May 28, 1875 he was promoted and received the title of a "Professor".

From 1876 onwards Kossak was docent for mathematics at the Bauakademie in Berlin, too, until in 1879 the Gewerbeakademie and the Bauakademie were united as TH Charlottenburg, where he stayed as professor for mathematics until his death on January 21, 1892. In the academic year 1883/84 he was head of the "Abteilung V für Allgemeine Wissenschaften" at this institution which amounted to the position of a dean of the department for the non-engineering sciences. And, finally, in 1885 Kossak had even improved his relation to Weierstraß and his students so far that his photo appeared in the photo album which was presented to Weierstraß on the occasion of his 70$^{\text{th}}$ birthday (Bölling, 1994, reproduction of vol. 1, p. 24, photo 7).

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