Arithmetic by manipulatives: a view through historical examples

Anne Rahn, Susanne Spies, University of Siegen, Department of Mathematics, Germany

Abstract

The use of manipulatives has become fundamental to the teaching of mathematics in primary schools. The materials used in mathematics education are surprisingly constant, even if the didactic guiding ideas have changed throughout history. The analysis of some still influential examples from the history of mathematics education since the 18th century on the basis of current didactic categories should serve to look at them from a new perspective and provide a framework for comparison.

Keywords: history of arithmetic, ICHMI, manipulatives, mathematics education, Anschauung

Introduction

The use of manipulatives is at least as old as arithmetic itself; just think of counting marks as notches on bones or psephoi-arithmetic in ancient Greece (cf. Wussing, 2008). Even if, according to Volkert (1986), the manipulatives lost their proof function with the "crisis of visualization" (Krise der Anschauung) in the 1870s, their influence on the teaching and learning of arithmetic remains unbroken. In this context, teaching materials are said to have at least an "epistemological guiding" function, to use Volkert's vocabulary.[1] The extent to which this guiding function is subject to change over of time is illustrated by selected examples from the history of mathematics education. In doing so, a view from the current didactic discussion is deliberately taken and criteria are developed on the basis of today's research, which allow a comparative description of the various examples. In this way, a classification including a meta-level is to be carried out for all the tools - an observation from the same point of view. In this way, the tools and the respective intentions of their inventors can be described in a multi-layered didactic field of tension, whereby a value judgement is of course inappropriate. However, from the perspective of the historical background, the selected tools can all be seen as innovative approaches according to their historical background. Since these are well-known and historically widely studied examples, the aim of this paper is not a

1 Volkert (1986) identifies three functions of Anschauung in the history of mathematics: an epistemological function, i.e. a proving function, an epistemological limiting function (only that which eliminates an intuitive or visual model exists) and an epistemological guiding function. The latter belongs to the field of heuristics and describes the discovery of structures by visualization, which can then be formally proven.

comprehensive historical account, but rather a comparative and exemplary description.

The Paper will therefore begin by discussing a possible definition of manipulative, on the basis of which the examples will be selected. Then, a system of categories is developed in order to derive the research questions for the analysis of the examples. The analysis is followed by a brief conclusion.

Manipulatives

From the early mathematical developments of mathematical aids, which included calculating with small stones or notches in bones, two branches of the development of materials for teaching mathematics can be identified. On the one hand, the invention of so-called calculating aids, ranging from simple calculating machines, such as Caspar Schott's calculating box or the abacus, to the pocket calculator. On the other hand, developments in developmental psychology have led to a rethinking of mathematics education (Moyer, 2001). The awareness of the cognitive development of learning according to Jean Piaget, especially the concrete operational stage, changed the view of teaching and learning (Piaget, 1970). In addition, Jerome Bruner's theory emphasized the different ways of representing an object of learning: "Any subject matter can be successfully taught to any child at any stage of development in an intellectually honest form" (Bruner, 1976, p.77). This development of didactics towards a constructivist view of learning led to the importance of working with hands-on materials until today.

In German-speaking countries, the use of the term ‘Arbeitsmittel’ is ambiguous. Definitions can be found that include all materials used in mathematics teaching (Schmidt-Thieme, 2015), as well as different categorizations depending on whether they are used by the teacher or the students. For example, there are classifications into ‘Anschauungsmittel’ (visual aids/ visualizations), which are used by the students, or ‘Veranschaulichungsmittel’ (illustrative aids), which are used by the teacher. Furthermore, a distinction is made between ‘Lösungshilfe’ (solution aids), ‘Lernhilfe’ (learning aids) and ‘Lerngegenstand’ (object of learning) (Krauthausen, 2018). As the assignment of a material to the definitions depends strongly on the situation in which it is used, this distinction is somewhat impractical for the analysis of historical examples. Hartshorn and Boren‘s definition of manipulatives clearly separates materials that are used as aids for reckoning from those that are intended to support the development of a mathematical concept:

> “In this Digest "manipulatives" will be understood to refer to objects that can be touched and moved by students to introduce or reinforce a mathematical concept.” (Hartshorn & Boren, 1990, p.2)

In this sense, manipulatives are objects that can be perceived with the external senses, that have a mathematical meaning and that can be handled in the broadest sense. These include two-dimensional representations (tables, diagrams, graphs) that can be pointed at and changed by drawing, as well as three-dimensional objects that can be moved in space. These are to be distinguished from those tools that are used purely as algorithmic calculation aids, such as marks for calculating on the lines. Since this function can also change in the course of history, we here will analyze materials that are explicitly introduced to build up a special arithmetical visualization (‘Anschauung’)[2] in mathematics teaching. The range of examples presented in the following begins with Johann Heinrich Pestalozzi as the first mathematics educator who emphasizesole of Anschauung in learning mathematics.

Selection of examples

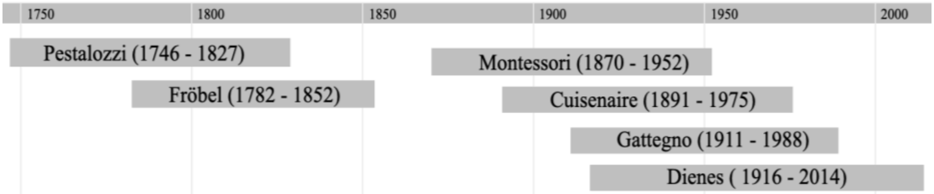

Fig.1 Timeline of the selected examples

In addition to the “Table of units” by Johann Heinrich Pestalozzi (1746-1827), the forefather of mathematical ‘Anschauung’ teaching arithmetics, the “Gifts and Occupations” of his student Friedrich Fröbel (1782-1852) will be presented. This is followed by the sensory materials of Maria Montessori (1870-1952) as a representative of reformist pedagogy. The materials of Zoltan Paul Dienes (1916-2014) and the rods of Georges Cuisenaire (1891-1975) will then be considered in the context of the development of New Mathematics. All these examples can be seen as manipulatives for arithmetic in the sense described above. In addition, the selected people and their materials provide a brief outline of the history of teaching arithmetic with manipulatives, and all of them still have an influence on current materials in mathematics teaching or are even still in use today.

Categories of description

Various criteria for evaluating working materials can be found in the literature (Krauthausen, 2018), but they mainly focus primarily on lesson planning. In this context, the mathematical learning content is usually only one sub-item among others. However, if the focus is placed on mathematical concept formation processes, these criteria no longer serve their purpose. What is needed is a more detailed analysis that focuses on different aspects of the process. Following Maier (1998), a distinction should be made here between formal and content-related criteria. If the formal level describes what is to be understood per se (Knowing WHAT), the nature of the understanding process can be described more precisely at the content level (Knowing HOW). In mathematics didactic discourse, these aspects are often discussed in isolation from each other, depending on the

2 For the sake of simplicity, the concept of mathematical Anschauung is considered very broadly here: In addition to the classical sensory perception of mathematical structures (empirical intuition), imaginations are included as well as pure intuition (reine Anschauung) in the Kantian sense or the figural concepts according to Fishbein (1993).

respective theoretical background. However, in order to provide the broadest possible classification of the historical examples, different perspectives on the use of manipulatives will be used in the following and brought together in a system of categories. Any contradictions that may arise from the different theoretical frameworks will be accepted for the time being and obvious interdependencies are will be ignored. Nor will there be any weighting of the criteria in terms of content.

Motivated by the subject-matter didactics of arithmetic, it must first be clarified which type of arithmetic 'Anschauung' the material is aimed at. Following Sirpinska's classification of visualizations, it can be asked whether the visualization is about the *representation of an object* and its details (figural) or whether the mathematical configuration is presented as a whole, focusing on the relationships between individual objects (structural) (Sirpinska, 2003). In the next step, it can be described HOW the didactic reference context for dealing with the objects is set. Three elements need to be considered: the linguistic support, the material handling and the pupils' mental activity. From a didactic point of view, the linguistic accompaniment of the instructional handling of working materials is of particular importance, since it is only through this that a mathematical meaning is assigned to the manipulative used (Maier&Schweiger, 1998). This accompaniment can consist of simple repetition and imitation of accompanying words, or it can be used flexibly in the sense of a meaningful visual language.

The enactive actions with the materials can be divided into purely mechanical, imitative actions or free exploration in the sense of learning by discovery (Winter, 2016). (pupil-centered vs. teacher-centered). The intended mental activity can then either be described as procedural operation with the mental images of the manipulatives, or the actions with the manipulatives also enable a conceptual insight into the mathematical structures, which then also provide, for example, arguments or proof ideas for generalization (see. e.g. Arcavi, 2003). Table 1 gives an overview of the emerging category system.

Table 1. System of Categories

What?	Structural	**Focus of intuition**	figural
How?	repetetiverepetitive	**Way of speaking**	meaningful
	Imitative	**Use of manipulatives**	self contained
	Procedural	**Mental activity**	conceptual

This categorization leads to the following research questions:

a) How can the original use of the manipulatives be described from the perspective of current mathematics didactics?
b) Can a development/pattern be identified over time?

c) From today's point of view, what possibilities do the manipulatives offer for teaching mathematics and for their targeted use?

Analysis of the examples

In order to answer the first question, the selected examples are first placed in the category system. In the course of the analysis based on the criteria, it should be noted that the historical examples presented here can only be presented in excerpts in this framework. The focus has been on arithmetic at the primary level.

Pestalozzis "Visual theory of number ratios"

The first example is the approach that was the first to give space to visualization in the teaching of mathematics - in contrast to the purely repetitive, language-based elementary teaching. The Swiss educator Johann-Heinrich Pestalozzi (1746-1827) is regarded as the forefather of a subject matter didactics that places the emphasis on 'Anschauung'. In particular, the fact that Anschauung is described as the "absolute foundation of all knowledge" and the "original form" of learning (cf. Pestalozzi, "Wie Gertrud ihre Kinder lehrt", 1870, p. 155) is still used to motivate work with manipulatives.

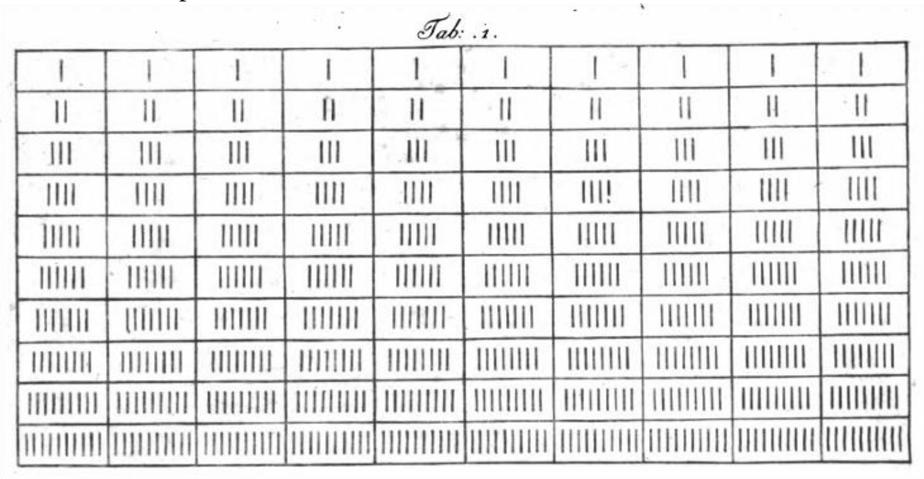

Fig.2: "Table of units"

According to Pestalozzi, the first experiences with elementary arithmetic should begin in early childhood with the help of uniform everyday objects and body parts, and the basic forms of number should be experienced as a composite set of uniform units. The naming of numbers with number words is then only regarded merely as an abbreviation (Pestalozzi, "Wie Gertrud ihre Kinder lehrt", 1870, p. 149). Thus, fingers become manipulatives in the above sense when they are given a mathematical content with number words. The aim of working with manipulatives is therefore first of all to build up a cardinal concept of numbers.

As the number of fingers is naturally limited, Pestalozzi adds further objects. He restricts himself to two-dimensional graphic means, that can be drawn by teachers and pupils, such as the divided straight line, the divided square for the formation of the concept of proportion and fractions, as he presents them in his "ABC der Anschauung" (1803) in a detailed practical instruction. Especially for the introduction of number relations and basic arithmetic, Pestalozzi suggests two-dimensional counting lines in "Die Anschauungslehre der Zahlverhältnisse" (1803), which are presented in a structured arrangement (Fig. 2).

The table is the basis for a whole series of arithmetic exercises to build up concepts of numbers, both ordinal and cardinal. According to Pestalozzi, the first thing to be experienced is unity and the "one and one in addition". The teacher is meant to demonstrate this by pointing to the first row of the table and a corresponding given text "1 times 1, 2 times 1, 3 times 1" etc. in exercise one. When moving to the next rows, two, three, etc. strokes are combined to form a unit. When moving to the next row, two, three, etc. strokes are combined to form a unit and the number of strokes is also determined. Here, thinking in units, called unitizing, is introduced. In exercise two, the bundled quantities are broken down and how they can be put together from the previous units and their proportions is worked out: For example: “5 times 1 is 2 times 2 and half of 2". At the same time, multiplication problems are presented as a summary of units of the same size, and the linguistically identical formulation of the multiplication tables is linked to a visualization. The pupil's task is first to listen attentively and then to imitate the text and gesture in response to questions. Care should also be taken to ensure that answers that can be given without gestures are also justified by the picture. (Cf. Pestalozzi, "Anschauungslehre der Zahlverhältnisse", p. 9Ff, translation by the authors).

Table 2. Pestalozzi

structural	**Focus of intuition**	figural
repetetive	**Way of speaking**	meaningful
imitative	**Use of manipulatives**	self contained
procedural	**Mental activity**	conceptual

By focusing on the division of numbers and the bundling or unbundling of units, the details of the individual natural numbers are brought into focus. The language used is characterized by precisely prescribed formulations, that must also be adhered to. The same applies to the actions on the diagram. Pestalozzi tries to get away from the schematic, memorized arithmetic of the so-called "tongue thrashing" and claims that arithmetic taught in this way is "only an exercise in reason, not at all a mere work of memory or a routine advantage of craftsmanship" (Pestalozzi, "Wie Gertrud ihre Kinder lehrt", p. 154). From today's point of view, the result of the method can rather be described as schematic or procedural, since the initial aim is to be able to demonstrate certain gestures and words by heart. Pestalozzi's proposal can therefore be placed in the proposed system of categories as shown in Table 2.[3]

Fröbelian „Legestäbchen“

The most striking difference between Johann Heinrich Pestalozzi and Friedrich Fröbel (1782-1852) has already been pointed out by Felix Klein, who emphasizes that Fröbel overcomes Pestalozzi's poverty of content by placing the three-dimensional figure at the beginning (Klein, 2013). In fact, his so-called Gifts and Occupations, which play a prominent role in his conception of the first kindergarten, consist of tangible materials. Among them is the laying of sticks, which can be interpreted as a three-dimensional materialization of Pestalozzi’s "Table of Units" (Fig. 1).

While the role of the gifts (especially nos. 1-6) for the formation of geometric-spatial skills is well documented from a current perspective (cf. Reinhold, Downton, & Livy, 2017) and has also been repeatedly linked to Fröbel's other professions (cf. e.g. Friedman & Alvis, 2021), their role for early arithmetic formation is rarely considered. It should be noted here that although the gifts are geometric shapes, which should also be used to gain initial mathematical experience and build up corresponding ideas, they are not intended as material exclusively for teaching mathematics. For example, the laying of chopsticks also begins by using the special properties of the chopstick (being straight) to represent everyday objects, followed by the laying of particularly beautiful patterns. And also with regard to the forms of cognition, for Fröbel form comes before number (Fröbel 1874, p. 572).

In his later reflections on the transition from kindergarten to primary-school, which he recorded in a letter to his pupil Emma Bothmann shortly before his death in 1852 under the title "Die Vermittlungsschule" (The Mediation School), Fröbel proposes the use of the chopsticks in two ways: Firstly, the chopsticks can be seen as ideal types of "similar objects" and thus replace the counting of body parts and everyday objects. In addition to the idea of quantity, experience can also be gained with number ratios, i.e. fractions. Secondly, Fröbel also wanted to use the sticks to learn the decimal position system by showing how ten sticks representing the unit are bundled into one representing the tens. In order for the children experience this, it is important for Fröbel to work with the 3-dimensional sticks and to replace ten of them by one in the next position. And on the other hand, similarly to the laying of life forms, to re-lay the numbers:

3 Note that the reason for this categorization is the way Pestalozzi foresees for dealing with the table in class, since a categorization is always only possible in the intended context. Of course, a more meaningful, self-contained or conceptual useage of the table would also be conceivable in another setting.

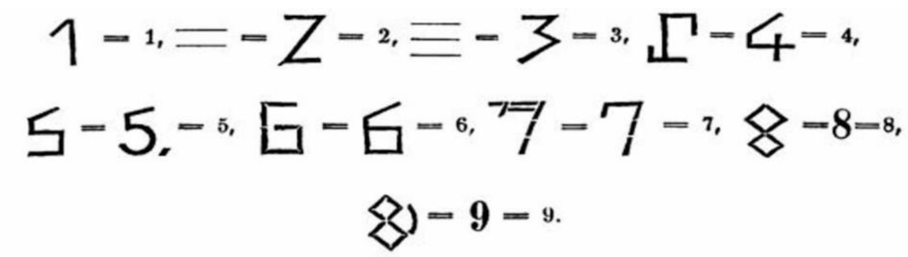

Fig.3 Numerals formed by chopsticks

Fröbel thus focuses on the details of individual numbers and the number line, rather than using the materials to explore general structures (which is of course also due to the intended place of use in pre-school).

In the conversations during the use of the materials, apart from songs and rhymes introduced by the kindergarten teacher as mnemonic sentences, there is no given language but the "accompanying word", which can be classified as a meaningful way of speaking.

Table 3. Fröbel

structural	**Focus of intuition**	figural
repetetive	**Way of speaking**	meaningful
imitative	**Use of manipulatives**	self contained
procedural	**Mental activity**	Conceptual

Schauwecker-Zimmer summarizes the work with the gifts as "activity pedagogy, not instruction pedagogy". It leads through "grasping", that is, perceiving with all senses via the "hunch" and the "accompanying word", to “understanding". (Schauwecker-Zimmer, 2012, p. 64). Thus, mental activity can also be described as conceptual.

Montessori's „Sinnesmaterial“

Maria Montessori's pedagogy is characterized as a child-centered pedagogy in which the child is to be self-active. "The child's mind is prepared for numbers not by certain "introductory ideas" hastily put forward by the teacher, but by an educational process, by a slow independent building up" (Montessori, 1928). The use of her materials takes place in a prepared environment, which can be described in terms of scaffolding. This type of environment guides the children's discoveries. The constant use of the same material enables the recognition of patterns and helps to focus attention when working independently. Montessori assumes that every child has an innate mathematical mind, to which she attributes the fundamental mathematical activities of ordering and comparing. She sees the child as a pattern recognizer, which leads to the fact that the given “sensory materials” leading to abstraction through ordering and comparing. She describes her initial sensory material as materialized abstraction.

> "Here was a sure guide: To help intelligence-building in children, objects isolating an abstraction were necessary" (Montessori, 1961, p. 134, trans. by author)

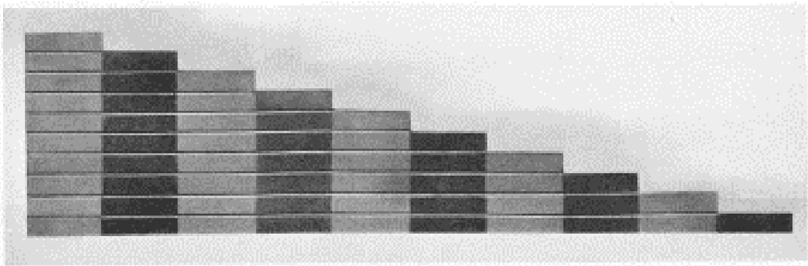

Abb. 6 und 7. Die breite und die lange Treppe

Fig. 4: big numerical sticks (Montessori, 1928, p. 41)

The first fundamental idea that the children should discover with the help of the sensory material is that of number, or a cardinal concept of number, through which the children should structure their preliminary experience of counting. "The children have all the instinctive knowledge that is necessary as a preparation for clear ideas in counting" (Montessori, 1928, p. 101). In working with the material, there is no need for verbalization on the part of the children, but language is given a place in the notation of their own knowledge and in the exchange between the children. Montessori describes this as "mental walks" in the prepared environment where each child can work with different materials and learn from each other.

Table 4. Montessori

structural	**Focus of intuition**	figural
repetetive	**Way of speaking**	meaningful
imitative	**Use of manipulatives**	self contained
procedural	**Mental activity**	conceptual

On this theoretical basis, the materials that provide the first access to arithmetic in terms of developing a concept of numbers are now classified. These include, among others, the number chips, the numerical sticks (Fig. 4) and the spindle box. At the formal level, the children are supposed to recognize mathematical structures by imitating the actions of the teachers at the content level and, if necessary, developing them further. They work primarily procedurally, but this does not completely exclude the development of conceptual knowledge. Their facilitated linguistic exchange in the mixed-age learning group is to be classified on a meaningful level through learning with and from each other. Linguistic instruction does not take place during the introduction to the material.

Cuisenaire rods

The Cuisenaire rods were designed by George Cuisenaire himself, but popularized by the educator Caleb Gattegno (1911–1988). Cuisenaire rods are coloured cuboids ranging from 1cm to 10cm in length, with the colours illustrating mathematical relationships (for example: two yellow rods equal orange). In 1960, the textbook "Les nombres en couleurs" was published in which Cuisenaire and Gattegno describe the various uses of the rods. Over time, a series of ten primary school textbooks was created. All the problems to be solved with the slide rules are based on an analytical approach, which allows and encourages different solutions on the part of the children (e.g. Fig. 5)

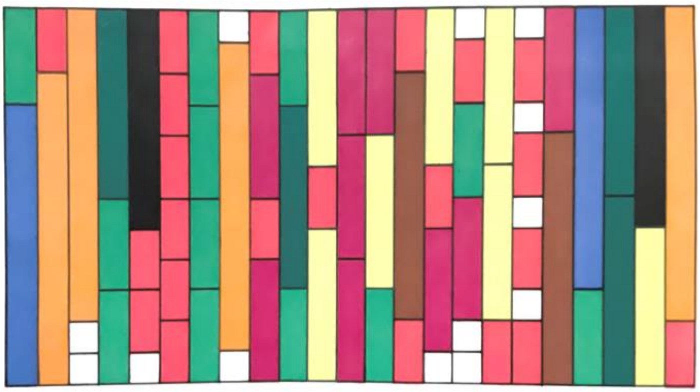

Fig. 5: The Pattern of twelve ("The Junior Numbers in Colour", p.2)

The didactic guidelines of the material can be summarized as follows: The learner is encouraged to pay attention to the relationships between the quantities. The recognition of these relationships is promoted through the use of mathematical vocabulary, symbols and notations, thus creating an "algebraic awareness" (Gattegno, 1983).

Table 5. Cuisenaire

structural	**Focus of intuition**	figural
repetetive	**Way of speaking**	meaningful
imitative	**Use of manipulatives**	self contained
procedural	**Mental activity**	conceptual

The emphasis is not on teaching arithmetic but on doing mathematics in a structural sense (Gattegno, 2010). For Cuisenaire, the problem to be solved was to find the means to prepare from the level of observation to the level of fixed and concrete fixation as a transition to abstraction. A book review published in 1985 states:

> "And this is what the Belgian inventor has achieved with his intuitive procedure of "coloured numbers", scientific and pedagogical, simple and attractive. (Alba, 1985, p.103, trans. By author)

This understanding of teaching mathematics with the help of the rods can be classified as follows. The aim is structural and is even called "algebra first". Its way of speaking is at a symbolic level through technical terms, the students work on tasks from so-called working cards, which build up a conceptual understanding on an algebraic level and not at an arithmetic level. For example, there are no divisions on the rods that would link them directly to a number, so that ideas of part-part, halving, doubling, and arithmetic operations can be thought of algebraically.

Dienes „multibase arithmetic blocks"

The last example is Zoltan Dienes, a representative of the "New Math". He saw this reform as necessary from an economic point of view, because in a changing world of work, arithmetic operations have taken a back seat to a general ability to think in structures. Dienes describes a six-stage model of constructive concept formation that starts with mathematical play. This is also his teaching method, which is carried out in several steps. Firstly, free play, secondly, structured play with the aim of discovering rules, thirdly playing with the rules in order to deepen them, and finally, there is free space for one's own ideas. The materials used are first of all uniform, everyday objects, in particular the logical blocks for set theory and the multi-system blocks. The latter does not only refer to base 10, but should also be used in other bases to understand the place value system and should "by no means be crammed in as pure arithmetic" (Hamann, 2018). Again, the emphasis is on the mathematical structure, which the children should discover independently through play, but above all understand conceptually. In the considerations of Dienes (1967), language only enters into the considerations of playing with materials in the sense of the mathematical formal language.

Table 6. Dienes

structural	**Focus of intuition**	figural
repetetive	**Way of speaking**	meaningful
imitative	**Use of manipulatives**	self contained
procedural	**Mental activity**	conceptual

"This mathematical language provides a useful, if not necessary, set of instruments" (Hamann, 2018, p.70, trans. By author). In this context, according to Dienes, language is not necessary to guide the action, so that no classification is possible here.

Conclusion

In a brief summary of the analysis, further research questions arising from the category system will now be addressed. In a synopsis, we first see that each example produces its own pattern according to our categories, which means that there is a specific didactic focus in each approach. Depending on the philosophical and didactic background, the context of the use of materials is very different. This can be seen, for example, in the comparison of Fröbel's and Cuisenaire's use of a very similar material made of sticks: Action and use depend on whether the focus is on building an ordinal or cardinal number concept or algebraic structures. This further illustrates that an empirical object does not speak for itself, but requires a theory-conscious use. Nor is there any evidence of a purposeful development towards a particular conception of mathematics education. Further analysis of manipulatives with the help of the category system can provide more information about the historical development, but also the analysis of new tools can make the didactic-philosophical background more precise and useful for classroom use.

In addition to the current issues of mathematics didactics (cf. Schmidt-Thieme & Weigand, 2015) in relation to the use of working materials, which are also reflected in our system of categories, it should be noted that the original way cannot simply be faded out. Rather, even from today's perspective, the respective advantages of the manipulatives still lie in their specific intended use. For example, it would be too short-sighted to use Cuisenaire's rods as an aid to calculate.

The different use of language to accompany the action is also interesting. There is a need for further research in this area, as well as in the question of the role of the enactive use of manipulatives in the construction of mental images or arithmetical intuition.

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