Operations with fractions in Spanish 16th century arithmetic texts

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Abstract

Measurement processes play a central role in the historical genesis of rational numbers. As a consequence, and even if the use and treatment of fractions differed among cultures, it is not surprising that this topic is already found in the oldest preserved mathematical texts. As expected, one of the fundamental aspects that was consistently covered by the authors throughout the history was related to the performance of the four basic operations (addition, subtraction, multiplication and division) with fractions. In this context, many different approaches and presentations can be identified among different texts. During the 16th century, many textbooks with arithmetic content were published in Spain. They were written by authors with different profiles, and pursuing different goals. Hence, it makes sense to try to identify the possibly different ways in which these authors approached the treatment of fractions in their books. More particularly, in this work we consider the treatment of the four basic operations with fractions in arithmetic books published in Spain around the 16th century. We describe the very different treatments that we have identified according to the number of addressed cases, and their ordering and we discuss the differences according to variables such as the number of elements, or the type and nature of the objects involved in the operation. This allows us to compare the behavior of the different authors, and opens the question about the underlying motivations for their choices, and also about their possible sources and mutual influences.

Keywords: fractions, arithmetic, operations, Spain, 16th century.

Introduction

The identification and recognition in ancient texts of what we call today 'fractions' can be a problematic task. Nevertheless, it is already possible to identify evidence of 'fractional measures' in the earliest known mathematical texts (Ritter, 1992). Thus, their presence and treatment in different cultures or by different authors is an interesting and lively topic from the point of view both of history of mathematics and of history of mathematics education.

From the point of view of history of mathematics, the study of fractions permits, for instance, the comparison and identification of similarities and differences among cultures (Chemla et al. 1992). It can also shed light to the close relation between culture and society and mathematics (Benoit, 1992). More recently, Moyon and Spiesser argue that their detailed analysis of the treatment of fractions in the Liber Abaci "permet de mieux éclairer l'oeuvre de Fibonacci dans son ensemble" (Moyon & Spiesser, 2015, p. 420).

From a more educational point of view, the different possible conceptions about the nature of fractions, the plurality of meanings and subconstructs associated to them, and the various algorithmic procedures that arise in practice

Bjarnadóttir, K., Furinghetti, F., Karp, A., Prytz, J., Schubring, G., Weiss, Y., & Zender, J. (Eds.) (2023). *"Dig where you stand" 7. Proceedings of the seventh International Conference on the History of Mathematics Education.* Münster: WTM-Verlag.

imply that the authors of books had to make a series of choices when presenting the material. The decisions made by the authors were probably motivated not only by their own beliefs, but also by their desire to create books that were useful for the potential learners. For example, Juan de Yciar, one of the authors that we will study, ended his presentation about the addition of fractions with the following words: "with this I end all that is related to the addition of fractions presented as short and succinctly as I could". Of course, different authors made different choices (Oller-Marcén, 2021) and it can be interesting to identify different profiles and underlying ideas.

In the late 15th century and mostly during 16th century there was a flourishing in Western Europe of arithmetic books intended to be "handbooks of applied mathematics, readily usable mathematics" (Swetz, 1992, p. 367). This was of course the consequence of the increasing and rich commercial activity. In this context, it is clear that fractions were a very relevant topic. The pedagogical approach adopted by the book authors could be an important feature for readers who probably wanted to learn in the easiest and fastest possible way. This surely had an impact on sales, and the authors made this point clear on their introductions (Muñoz-Escolano & Oller-Marcén, 2020).

Spain was no exception to this phenomenon (Madrid et al., 2020), so in this situation the following research question naturally arises: how were fractions introduced and treated in 16th century Spanish arithmetic texts? As far as we know, this question has not been thoroughly addressed in the existing research, so we aim to fill this gap in the literature. In this paper, we narrow our interest and we focus on the treatment of the four basic operations with fractions. In particular we have the following specific research objectives:

- To describe the different cases considered by the authors when introducing the four basic operations with fractions.
- To determine the variables used by the authors in order to organize and sequence their presentations.
- To give possible interpretations to the authors' choices and decisions.

The paper is organized as follows. In the next section we provide some context and background with a brief overview about 16th century Spanish arithmetic texts. Then, we introduce the theoretical and methodological framework for our study. Next, we present the main results, and we close the paper with some final remarks.

Background.

The 16th century in Spain is usually considered to be the "Golden Age" of arts and literature. However, the situation regarding sciences and mathematics was traditionally considered to be much worse. One of the most famous critiques was that of José de Echegaray, mathematician and Nobel Prize of literature, who lamented that Spain was "a country without science" (Echegaray, 1866, p. 27).

The first serious attempts to place value on Spanish science and mathematics authors from the 16th century began with the seminal works by Picatoste (1891), Fernández Vallín (1893), and Rey Pastor (1926). Although both Picatoste and Fernández Vallín were mathematicians, they focused on all sciences and they provided quite extensive dictionaries of Spanish scientific authors, giving some brief biographical and bibliographical information about them.

The book by Rey Pastor, "*The Spanish mathematicians of the 16th century*", tried to give an overview of the 16th century Spanish mathematicians known at the time when it was written (early 20th century). Even if a bit naïve, Rey Pastor attempts a first classification of authors into geometers, arithmeticians, and algebraists. As arithmeticians, he mentions Pedro Sánchez Ciruelo, Juan Martinez Silíceo, Juan de Ortega, and Álvaro Tomás. As algebraists, he introduces Marco Aurel, Juan Pérez de Moya, Antich Rocha, and Pedro Núñez. Finally, as geometers, he mentions Molina Cano and Jaime Falcó. In some cases, Rey Pastor goes into details, describes the mathematical content of some of the books by these authors and discusses their relative importance.

In any case, the previously mentioned traditional negative view about Spanish mathematics during the 16th century has been revised, and in many ways overcome. It is true that in that time "scientific work was only stimulated due to its usefulness" (Salavert Fabiani, 1995, p. 253). But even in this context, López Piñero (1979) reports the publication of 77 first editions of scientific books in Spain during the period 1475-1600. In the case of mathematics, it is noteworthy that printed arithmetic textbooks arrived in the Iberian Peninsula quite early, especially in the context of the Crown of Aragón. The *Treviso Arithmetic* dates from 1478 (Smith, 1924), while the *Suma de la art de arismetica* by Francesc de Santcliment was printed in Catalan in Barcelona only five year later, in 1482. This book was soon translated into Spanish by the same author in a slightly revised version printed in Zaragoza in 1486 (Escobedo, 2010). Also, in Latin, we can mention the *Tractatus arithmeticae practice* by the Aragonese Pedro Sánchez Ciruelo, printed in Paris in 1495.

This relatively early introduction of arithmetic in the Spanish context led to a rather big number of publications. Thus, between 1482 and 1600, Salavert Fabiani (1994) has identified up to 47 authors, 58 different books, and a total of 106 editions of arithmetic books. In his study, according to the goals and to the context of publication, Salavert Fabiani distinguishes between university and practical arithmetic textbooks.

University arithmetic textbooks were essentially academic. They were mostly devoted to studying integer numbers and magnitudes, and had a clear propaedeutic character towards music, philosophy, etc. They were mostly written in Latin and almost all of them were printed in Paris, where their authors were teaching.

On the other hand, practical arithmetic textbooks aimed to providing the tools required to solve the problems arising in commercial settings and hence included a wider range of topics (Salavert Fabiani, 1990). These books were almost always written in vernacular, and their places of edition were much more geographically disperse.

From a quantitative point of view, practical arithmetic textbooks outnumbered university arithmetic textbooks in global terms. However, the situation varied over time. From 1481 to 1525 university arithmetic textbooks were more numerous, and the situation inverted after then. This is consistent with the fact that during the 16th century most of the teaching of mathematics in Spain took place outside universities (Madrid, 2016).

Finally, it must be noted that many of the practical arithmetic textbooks published during this century had a great impact and influence. Some of them were still being printed and circulating after more than 100 years (Oller-Marcén, 2020). In fact, this might partly explain the relative stagnation in the production of original mathematics books during the 17th century (Navarro Brotons, 1996).

Theoretical and methodological considerations

From a historical point of view, the genesis of fractions is directly associated to the process of measuring which, according to Bishop (1991, p. 34) is one of the "universal and significant activities for the development of mathematical ideas". This process can be approached either by direct measurement or by the comparison of quantities, with each possibility leading to its own conceptual and procedural framework (Lamon, 2020). As a consequence of these universality, the use and treatment of fractions differed among cultures (Chemla, 1994).

Very often objects such as integers, proper fractions (those smaller than 1), improper fractions (greater than one, but written in fractional form), mixed numbers, or fractions of a fraction were conceived as different concepts from a purely epistemological point of view. These conceptions arise from the different possible actions leading to the notion of fraction, and are also related to the different subconstructs, or meanings, associated to fractions (Behr et al., 1992). In this regard, and in the context of medieval Arab texts, Djebbar (1992, p. 229) points out that "les entiers, les fractions, les sommes d'entiers et fractions et les sommes de fractions sont des entités distinctes, les auteurs consacrent ainsi a la multiplication et a la division autant de sections qu'il y a de combinaisons, deux à deux, de ces entités".

The different epistemological positions permeated the algorithmic processes. For instance, Chace and Manning (1927), in their introduction to the *Rhind Mathematical Papyrus*, show how Egyptian algorithms for operations with fractions were heavily influenced by the type of objects that they took into consideration (integers, 2/3, and fractions with numerator 1), and by certain other restrictions such as "that the fractions in an expression should all be different" (p. 8).

Finally, the close relation of fractions to measuring also had implications over procedural aspects. Thus, the Chinese algorithm for the addition or subtraction of fractions given in the *Jiuzhang suanshu* does not involve the requirement of reducing the fractions to the least common denominator (Chemla & Guo, 2004, p. 132), probably because imposing a minimality condition to the common denominator is not a requirement in a real measurement process.

We approached documentary research in the sense of (McCulloch, 2004) focused on the content of the analyzed documents (Prior, 2016, p. 172). In particular we are dealing with historical mathematics textbooks, a valuable source for the history of mathematics education (Schubring, 2023). According to Scott (1990) the documents used in documentary research must be assessed against four criteria: authenticity, credibility, representativeness, and meaning. In order to ensure that these four criteria were met, and taking into account our research goals, we introduced some restrictions when selecting our sources.

Table 1. Analyzed books

Author	Title	Place and date of 1st ed.	Lang.
Francesc de Santcliment	*Suma de la art de arismetica*	Barcelona - 1482	Catalan
Juan de Ortega	*Compusicion de la arte dela arismetica y juntamente de geometria*	Lyon - 1512	Spanish
Juan Andres	*Sumario breue de la pratica dela arithmetica*	Valencia - 1515	Spanish
Joan Ventallol	*Pratica mercantiuol*	Mallorca - 1521	Catalan
Gaspar de Texeda	*Suma de arithmetica practica y de todas mercaderias con la horden de contadores*	Valladolid - 1546	Spanish
Juan de Yciar	*Arithmetica practica*	Zaragoza - 1549	Spanish
Marco Aurel	*Libro primero, de arithmetica algebratica*	Valencia - 1552	Spanish
Juan Perez de Moya	*Arithmetica practica, y specvlatiua*	Salamanca - 1562	Spanish
Antich Rocha	*Arithmetica*	Barcelona - 1564	Spanish
Miguel Geronimo de Santa Cruz	*Libro de arithmetica especulativa, y practica, intitulado el dorado contador*	Madrid - 1594	Spanish
Bernat Vila	*Reglas Brevs de arithmetica*	Barcelona - 1596	Catalan
Geronymo Cortes	*Arithmetica practica*	Valencia - 1604	Spanish

All the selected textbooks are important from the point of view of history of mathematics in Spain for different reasons. They were also important during their time and many of them circulated for decades (and even centuries) after they publication, having in some cases an important influence over later works. As an example, the books by Perez de Moya or by Santa Cruz were still being reprinted in the last decade of the 18th century (Valladares Reguero, 1997; Madrid et al., 2015).

We have analyzed the first edition of these books with only two exceptions. In the case of the book by Santa Cruz we had to use the second edition published in Sevilla in 1603, and in the case of the book by Cortes we used a later edition from 1659. In both cases it is known that they were reprints with no differences in content with respect to the first edition. In all cases we have been able to access publicly available digitized versions of the books.

In order to carry out our analysis, we focused on the chapters, or sections, of the books devoted to fractions. More particularly, in those devoted to the presentation of the algorithms for the four basic operations: addition, subtraction, multiplication, and division. During our research we have taken into consideration three variables for each of the four operations:

- The number of different cases considered by the author.
- The defining characteristics of the cases.
- The considered cases, and the order in which they were presented.

Finally, we were also interested in describing the interrelations between these variables in order to understand some of the underlying authors' decisions.

On the number of cases

We first focus on the number of different cases considered by the authors for each of the four basic operations. Table 2 provides a summary of these information.

Table 2. Number of cases for each operation

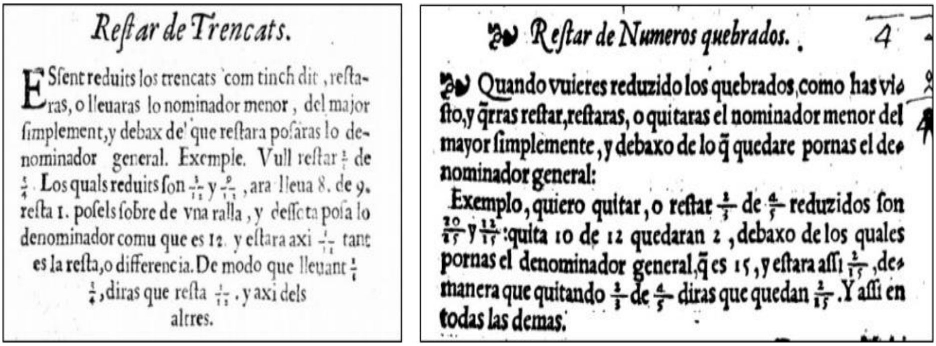
Author	Addition	Subtraction	Multiplication	Division
Francesc de Santcliment	1	7	5	6
Juan de Ortega	9	11	7	7
Juan Andres	5	5	5	5
Joan Ventallol	6	5	6	6
Gaspar de Texeda	1	5	1	3
Juan de Yciar	3	4	4	3
Marco Aurel	1	1	1	1
Juan Perez de Moya	6	5	5	5
Antich Rocha	1	1	4	4
Miguel Geronimo de Santa Cruz	3	4	4	3
Bernat Vila	1	1	1	1
Geronymo Cortes	4	4	5	6

The authors themselves used different terminologies in their discourse when introducing the different cases. For example, Ventallol uses *retgla* (rule), Pérez de Moya uses *differencia* (distinction), and Texeda uses *modo* (way). Some authors just enumerated a list of different examples without further explanation. However, all the considered authors present the different cases always as solved exercises.

It is noteworthy that only two pairs of authors coincide with respect to the number of cases that they take into account. First, we have Marco Aurel and Bernat Vila. They are the only authors that consider just one case for each operation. These two authors provide a unified approach consisting on the following algorithms:

- For addition, subtraction and division: reduce the involved fractions to a common denominator, and then perform the operation with the resulting numerators.
- For multiplication: multiply numerators and denominators.

The coincidence between these two authors is explained if we have a closer look at the text. In fact (see Figure 1) Vila's presentation about fractions seems to be an almost verbatim translation of the book by Marco Aurel in which only the numerical examples differ.



Reſtar de Trencats.

ESſent reduits los trencats com tinch dit, reſtaras, o lleuaras lo nominador menor, del major ſimplement, y debax de que reſtara poſaras lo denominador general. Exemple. Vull reſtar $\frac{2}{3}$ de $\frac{3}{4}$. Los quals reduits ſon $\frac{8}{12}$ y $\frac{9}{12}$, ara lleua 8. de 9. reſta 1. poſels ſobre de vna ralla, y deſſcta poſa lo denominador comu que es 12. y eſtara axi $\frac{1}{12}$ tant es la reſta, o differencia. De modo que lleuant $\frac{2}{3}$ $\frac{3}{4}$, diras que reſta $\frac{1}{12}$. y axi dels altres.

Reſtar de Numeros quebrados. 4

Quando vuieres reduzido los quebrados, como has viſto, y q̃rras reſtar, reſtaras, o quitaras el nominador menor del mayor ſimplemente, y debaxo de lo q̃ quedare pornas el denominador general:

Exemplo, quiero quitar, o reſtar $\frac{2}{3}$ de $\frac{4}{5}$ reduzidos ſon $\frac{10}{15}$ y $\frac{12}{15}$: quita 10 de 12 quedaran 2, debaxo de los quales pornas el denominador general, q̃ es 15, y eſtara aſſi $\frac{2}{15}$, de manera que quitando $\frac{2}{3}$ de $\frac{4}{5}$ diras que quedan $\frac{2}{15}$. Y aſſi en todas las demas.

Fig. 1. Subtraction of fractions by Vila (left) and Marco Aurel (right)[1]

Then, we have Juan de Yciar and Miguel Geronimo de Santa Cruz. These authors behave in the same way with respect to the number of cases but, as we will see in the next section, the defining characteristics of the cases were different. In addition to the wide variability among the authors, another interesting feature illustrated by Table 2 is that only Juan Andres considered the same number of cases for all the operations. In fact, as we will see later, he considered essentially the same five cases in the same order. All the remaining authors approach each operation separately. This is not surprising if we take into account the different particular meanings of each operation, as well as restrictions like the impossibility of obtaining negative results

In any case, it is difficult to identify trends in the data from Table 2. We see that most of the authors (Andres, Ventallol, Yciar, Aurel, Pérez de Moya, Santa Cruz, Vila and Cortes) try to consider a similar number of cases in each operation, with no or small variations. This might indicate an attempt of systematicity, which is not always fulfilled for reasons that will be suggested in the next sections. The remaining authors behave more erratically, especially Juan de Ortega, who always considered the greatest number of cases, and behaved differently from every other author in all the four operations. This particularity will lead us to have a closer look at his work in a forthcoming section.

[1] Once you have reduced the fractions [to common denominator], as you have seen, if you want to subtract them, you will just subtract the smaller numerator from the bigger one, and under what remains you will place the common denominator. (Translation by the author).

On the defining characteristics of the cases

As we have seen when addressing the similarities between Juan de Yciar and Miguel Geronimo de Santa Cruz, if we want to have a better understanding of the situation, we cannot rely on the number of cases alone. If we look at the characteristics used by the authors in order to define and describe the cases that they were considering, we see that they were essentially three:

- N. Number of objects involved in the operation: two or more.
- D. Denominators of the objects involved in the operation: equal or not.
- T. Type of objects involved in the operation: integers, proper fractions, mixed numbers, etc.

In Table 3 we show which of these characteristics were taken into consideration by the authors. Of course, when an author considered only one case for a certain operation, there is no relevant information to give.

Table 3. Characteristics considered when defining cases

Author	Defining characteristics			
	Addition	Subtraction	Multiplication	Division
Francesc de Santcliment		T	T	T
Juan de Ortega	N, D, T	N, T	T	T
Juan Andres	T	T	T	T
Joan Ventallol	N, D, T	T	T	T
Gaspar de Texeda		D, T		T
Juan de Yciar	N, T	T	T	T
Marco Aurel				
Juan Perez de Moya	N, T	T	T	T
Antich Rocha			T	T
Miguel Geronimo de Santa Cruz	D, T	T	T	T
Bernat Vila				
Geronymo Cortes	N, D, T	T	T	T

First of all, observe that the type of objects plays a role in all the operations and for every author. However, the situation is quite different depending on the operation. Generally speaking, addition is the operation involving the greatest number of characteristics, while for multiplication and division the type of objects is the only characteristic defining the cases. In fact, we could argue that this is also the case for subtraction, since only two authors consider other characteristics.

The number of objects is a characteristic that is present almost exclusively in addition. There might be several explanations for this phenomenon, but we think that it can be mainly related to the fact that it is easier to conceive addition as an *n*-ary operation for $n > 2$. In the case of multiplication there might be semantic difficulties when trying to associate a meaning to the product of more than two concrete quantities, while for subtraction and division there are also issues related to their non-commutativity and non-associativity (even for abstract numbers) that could have prevented authors to consider the number of objects as a relevant defining characteristic of the cases.

The same situation holds for the remaining characteristic, the denominator of the involved objects. In this case its presence is probably explained by the requirement to reduce to a common denominator the given summands, which sometimes leads to the consideration of fractions with the same denominator as a particular case.

The consideration of the characteristics defining the cases allows us to discriminate authors that seemed to behave in the same fashion. This was the case of Juan de Yciar and Miguel Geronimo de Santa Cruz who, in spite of considering the same number of cases for every operation, use different characteristics to define them for addition. We will soon see that they also differ in the particular cases that they consider and also in their ordering.

On the cases and their ordering

So far, we have described the number of cases and their defining characteristics. These elements have already highlighted the many different approaches adopted by the authors of the books. In fact, we can already say that only two authors behave exactly the same: Marco Aurel and Bernat Vila. All the remaining authors differ globally either in the number of cases or in the characteristics that they use to describe them.However, if we go deeper in our analysis and have a close look at the cases and their ordering, we will see that there are even more interesting differences that might provide some clues about the motivations and the didactical choices made by the authors.

Table 4. Cases considered by Juan de Yciar and Miguel Geronimo de Santa Cruz for addition.

Author	Cases and ordering
Juan de Yciar	1º proper fraction + proper fraction 2º mixed number + mixed number 3º more than two summands
Miguel Geronimo de Santa Cruz	1º proper fractions with equal denominators 2º proper fractions with different denominators 3º mixed numbers

For example, in Table 4 we show the three cases that Juan de Yciar and Miguel Geronimo de Santa Cruz considered when dealing with addition. We have already pointed out in the previous section that the defining characteristics were different. As we see both authors adopt very different approaches.

Yciar begins by showing how to add two fractions. He states that "this rule involves reducing [to common denominator] and adding" (fo. 34r), giving two examples with different denominators and making no comment about the trivial case of equal denominators. Then, he proceeds to show how to add two mixed numbers, for which he proposes to work with the integer and fractional part separately. Finally, he extends the given ideas, without many explanations, to the case of an arbitrary number of summands.

Table 5. Cases considered by Santcliment, Andres, Perez de Moya and Cortes for multiplication.

Author	Cases and ordering
Francesc de Santcliment	1º proper fraction × proper fraction 2º mixed number × proper fraction 3º mixed number × mixed number 4º integer × mixed number 5º integer × proper fraction
Juan Andres	1º proper fraction × proper fraction 2º proper fraction × mixed number 3º proper fraction × integer 4º mixed number × mixed number 5º mixed number × integer
Juan Perez de Moya	1º proper fraction × proper fraction 2º integer × proper fraction 3º integer × mixed number 4º mixed number × proper fraction 5º mixed number × mixed number
Geronymo Cortes	1º proper fraction × proper fraction 2º integer × proper fraction 3º mixed number × proper fraction 4º mixed number × mixed number 5º fraction of fraction × fraction of fraction

Santa Cruz, on the other hand, begins by showing how to add any number of proper fractions with the same denominator which "are easy to add, since adding up the numerators and dividing the sum by the common denominator gives the sought result". Then he moves to the case of different denominators, which he reduces to the previous one in the obvious way. He ends his presentation by addressing the case of mixed numbers in the same way as Yciar.

So, both authors seem to progress in an increasing order of perceived difficulty but they seem to disagree in the origin of the difficulty. For Yciar, difficulty is not related with the equality or not of the denominators, since that characteristic is not mentioned at all. For Santa Cruz, however, the irrelevant characteristic is the number of summands.

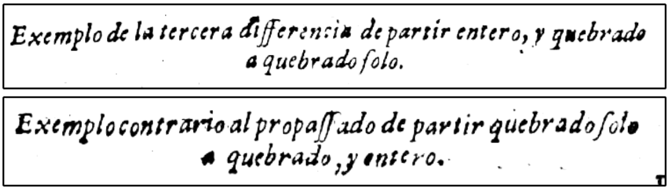
Both authors seem to consider that mixed numbers are more difficult to deal with than proper fractions. This was to be expected and, in fact, is the case for all authors. Also, all those authors that took into account the number of summands obviously considered that the case of two summands was the easiest. However, the agreement was not complete regarding the treatment of the denominators.

As seen in Table 3, five authors considered the equality or not of denominators as a relevant characteristic to define their cases. We can identify two different positions regarding this. On one hand, we have Ventallol and Santa Cruz, who presented the case of equal denominators as the first and easiest one. On the other hand, Ortega, Texeda and Cortes adopt the opposite position, first giving a general rule for different denominators and presenting the case of equal denominators just as an easier particular case of the given rule.

These options are made explicit by the authors themselves. For instance, Santa Cruz begins his discussion on the addition of fractions with the following words: "Adding two or more fractions having the same denominator is easy" (fo. 55r). Cortes, in turn, ends his presentation saying that "when the fractions that have to be added have the same denominator, it is not necessary to apply the rule, just add the numerators and divide by the denominator of the fractions" (p. 169).

We can further illustrate the different approaches made by the authors with another example. If we focus on multiplication, we see (Tables 2 and 3) that Santcliment, Andres, Perez de Moya and Cortes consider the same number of cases defined by the same characteristic. However, a closer look (Table 5) reveals important differences among them.

There are two interesting points to make in the light of Table 5. First, we see that Geronymo Cortes takes into account objects (fractions of fractions, like ½ of ¾) that are not present in the other authors' discourse. In fact, this is not exactly what is happening. Perez de Moya, for example, did cover such 'fractions of fractions' in his book, but in a completely different chapter solely devoted to them. Cortes, however, seemed to consider that these objects did not deserve to be separated from fractions.


Exemplo de la tercera differencia de partir entero, y quebrado a quebrado ſolo.

Exemplo contrario al propaſſado de partir quebrado ſolo a quebrado, y entero.

Fig. 2. Dividing a mixed number by a proper fraction (above) and vice versa (below) in the book by Geronymo Cortes (p. 182)

Second, note that Santcliment, Andres and Perez de Moya present the exact same five cases. This is not a coincidence. If we consider proper fractions, integers and mixed numbers, commutativity implies that there are six possible multiplications. Since the multiplication of two integers can be taken for granted, this leaves us with the five given cases. However, even if these authors present the same cases, they order them differently.

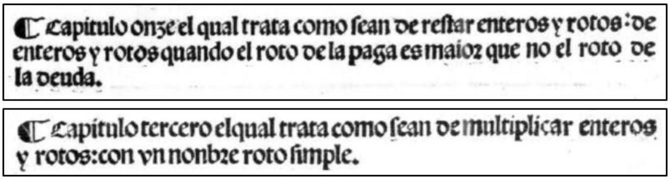
About this, it is also worth noting that, as we already pointed out, Juan Andres considered these five cases, in the same order, for all the four operations. However, this brings to the table the issue of the non-commutativity of subtraction and division, which was very seldom addressed by the authors. For example, when dealing with addition, the second case introduced by Andres was 'proper fraction

+ mixed number', while for subtraction the second case was 'mixed number – proper fraction'. Clearly Andres chose this order because otherwise the result would be negative (which was impossible). However, when dealing with subtraction the second case was 'proper fraction ÷ mixed number' and the opposite possibility was omitted without explanation even if it makes sense (Cortes, for instance, considers that possibility, see Figure 2).

An "outlier". Juan de Ortega

Ortega's *Compusicion de la arte dela arismetica y juntamente de geometria* was a very important book whose translation into French from 1515 was, in fact, the first commercial arithmetic printed in France. Due to its many reeditions before the end of the 16th century in Spain, France, and Italy (Rey Pastor, 1926), we think that its potential international influence further justifies our interest. In Tables 2 and 3 the many particularities of this text, which justify a closer look at it.

It is clear that Ortega considered that each case required an individual treatment. We know this because for each operation he arranged and structured the text in different "chapters" (sections, in fact) corresponding to the different cases. In Figure 3 we give the heading of two such chapters.


Capitulo onze el qual trata como sean de restar enteros y rotos: de enteros y rotos quando el roto de la paga es maior que no el roto de la deuda.

Capitulo tercero el qual trata como sean de multiplicar enteros y rotos: con vn nonbre roto simple.

Fig. 3. Headings for case 11º of subtraction (above) and case 3º of multiplication (below) in the book by Juan de Ortega (fo. 54r-55r)

Table 6 shows the different cases considered by Juan de Ortega for each of the four basic operations. As we already mentioned, for multiplication and division the only characteristic that defines the cases is the type of objects. The higher number of cases in comparison to the other authors is explained because Ortega considers fractions of fractions as relevant objects. For addition and subtraction, the treatment by Ortega is much more confuse, considering characteristics and cases that are not to be found in any other of the analyzed authors. The treatment of subtraction is particularly cumbersome, since Ortega also takes into consideration the possibility or not of directly subtracting the fractional parts when dealing with mixed numbers (see cases 10º and 11º).

Table 6. Cases considered by Juan de Ortega for each operation.

Addition	Subtraction
1º proper fraction + proper fraction, when the result is a proper fraction 2º proper fraction + proper fraction, when the result is a mixed number	1º proper fraction – proper fraction 2º proper fraction – sum of several proper fractions

3º more than two proper fractions 4º fraction of fraction + proper fraction 5º fraction of fraction + fraction of fraction 6º fraction of mixed number + proper fraction 7º mixed number + mixed number 8º more than two mixed numbers 9º more than two proper fractions with equal denominators	3º sum of several proper fractions – proper fraction 4º sum of two proper fractions – sum of two proper fractions 5º mixed number – sum of several proper fractions 6º sum of several proper fractions – integer 7º mixed number – integer 8º integer – mixed number 9º sum of mixed number and proper fraction – sum of mixed number and proper fraction 10º mixed number – mixed number, when the first fraction is greater than the second 11º mixed number – mixed number, when the first fraction is smaller than the second
Multiplication	Division
1º proper fraction × proper fraction 2º integer × proper fraction 3º mixed number × proper fraction 4º mixed number × mixed number 5º integer × mixed number 6º fraction of fraction × proper fraction 7º fraction of fraction × fraction of fraction	1º proper fraction ÷ proper fraction 2º fraction of fraction ÷ proper fraction 3º fraction of fraction ÷ fraction of fraction 4º integer ÷ proper fraction 5º mixed number ÷ mixed number 6º integer ÷ mixed number 7º mixed number ÷ proper fraction

Some concluding remarks

We have given a detailed view of the treatment of the four basic operations with fractions in twelve important 16th century Spanish arithmetic books. Our results provide an interesting example of how ancient textbooks authors dealt with some epistemological difficulties related to the important concept of fraction. All of them felt the necessity to consider different cases in the context of the four basic arithmetic operations, but we see how they did not agree on the treatment of what we would call today didactic variables (Artigue, 1992). This shows, in particular, that the concern for the selection and sequencing of examples and problems (Karp, 2015) is timeless.

Only two authors (Marco Aurel and Bernat Vila) agreed in not considering several different individual cases. The consideration of many cases according to different characteristics was not uncommon at the time, with a paradigmatic example being the resolution of algebraic equations (Massa-Esteve, 2012). However, it was rather surprising to find such a wide variation among authors in a much more elementary topic. Putting aside the aforementioned Marco Aurel and

Vernat Vila, the remaining ten authors behaved in a different way. It is difficult to formulate hypotheses for some of the reasons behind the authors' choices. In the case of Aurel, it could be argued that the ultimate goal of the book was to introduce Algebra, so that elementary arithmetic was treated in less detail, with a more general approach requiring just one case per operation. Also, books directly addressed to merchants, with a more practical approach, were more discursive in their presentation, paying attention to a higher number of particular cases, and giving much more importance to semantic considerations.

Unlike in the case of algebra (Romero-Vallhonesta, 2011), the very few similarities identified among the books makes it quite difficult to explore possible influences among authors, as well as to identify common sources in the treatment of fractions. In any case, we have shown that Bernat Vila almost completely borrowed his material on this topic from the book by Marco Aurel. As far as we know Vila's book has not been carefully investigated and its similarities with Aurel's work had not yet been identified. This opens the door to further research. As for possible sources, we already pointed out that Djebbar (1992) notes that the consideration of several cases in the operations with fractions was common in the Arab medieval context. The only author who did not follow that approach was Marco Aurel, a German whose main sources are known to be of a different origin (Romero-Vallhonesta & Massa-Esteve, 2018). Our work suggests an interesting line of research related to the search for possible sources and influences regarding the topic of fractions.

As further work, it would be interesting to have a closer look at the procedures used in the different cases, presenting them in detail, and possibly relating them to the use of abstract numbers, concrete quantities, or both. Finally, it would also be interesting to study the evolution of the treatment of the operations with fractions over time since different cases were still considered during the 18th or 19th centuries (Zubiaur, 1718; Gavilán Reyes, 1897).

Acknowledgment. Research group "Investigación en Educación Matemática" (S60_23R).

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