Making Cauchy's *Cours d'analyse* "elementary": Dirichlet's Lectures on Series

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Abstract

In the period between ca. 1820 and 1860, Cauchy's theories on real analysis met with a mixed reception in Germany. On the whole, however, they significantly influenced the development toward modern analysis. In this paper contributions of Peter Gustav Lejeune Dirichlet, who fostered this process of transition through his teaching, are investigated, mainly with regard to his adaption of a part of Cauchy's Cours d'analyse *in a lecture course on series from winter semester 1840/41. In order to attain a better understanding for his specific didactic approach, Dirichlet's course is compared with Schlömilch's* Handbuch der algebraischen Analysis *(1845).*

Keywords: Cauchy, Dirichlet, Schlömilch, series, rearrangement, elementary exposition

Introduction

Perhaps Cauchy's most influential textbook on the foundations of real – and to some extent also complex – analysis was his *Cours d'analyse* from 1821 (part 1 on algebraic analysis, the second part never appeared). This work met with a mixed reception in Germany (Reich, 2003). On the one hand, a German translation of the *Cours* by C.L.B. Huzler appeared as early as 1828. On the other hand, Cauchy's book – as well as its translation – was heavily criticized, mainly for its ostensibly too intricate exposition of fundamental notions. Critics such as Johann Tobias Mayer and Martin Ohm, as well as the editor of the German version, Huzler, were not in the first rank of German mathematicians at the time. Fortunately, however, we have an example for the adaption of Cauchy's theory by a first rank mathematician, Peter Gustav Lejeune Dirichlet (1805–1859). In WS 1840/41 he gave a course at the University of Berlin on "Elements of Series," which corresponded to the relevant and especially important chapter VI of Cauchy's *Cours*. From lecture notes of this course taken by Ludwig Seidel, later professor at the University of Munich, we can clearly discern the contents on which Dirichlet concentrated, and we can gain an insight into his particular approach to teaching "elementary" subjects.

Of course, Dirichlet was not the only one who intended to combine specific didactic goals with an exposition of Cauchy's analysis. Another, rather prominent example was Oskar Schlömilch's *Handbuch der algebraischen Analysis* (1845), which provides an interesting contrast to Dirichlet's approach.

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Dirichlet and his Course on "Elements of Series"

Originally, Dirichlet's family came from a French speaking area of Belgium near the German border. The epithet "Lejeune" = "the younger" had already been adopted by Dirichlet's great-grandfather. Dirichlet was born in Düren, in the neighborhood of Aachen, but instead of beginning his studies at a nearby university, he studied in Paris from 1822 to 1826. As a foreigner, he was not admitted to the École Polytechnique, where Poisson and Cauchy taught, but he became acquainted with both, and he studied their works very intensively (it may even be that Dirichlet did attend courses given by Cauchy at the Collège de France). From 1829 to 1855 he was professor at the University and the Military Academy in Berlin, and in 1855 he became successor of Gauß in Göttingen; for biographic details see (Merzbach, 2018). Dirichlet is very well renowned for his contributions to number theory, Fourier analysis, integral calculus, and mathematical physics including potential theory. Several mathematicians who had a significant impact on the reorganization of mathematics and its teaching during the second half of the 19th century, had studied with Dirichlet. Perhaps the best-known of these are Dedekind, Riemann, Kronecker, Lipschitz, Heine, Christoffel, and Seidel.

Dirichlet's favorite teaching subjects at the university were number theory, integral calculus, partial differential equations together with potential theory, and probability theory; for the latter field see (Fischer, 1994). In most cases, Dirichlet's lectures were addressed to advanced students. The course we will investigate in the following was an exception. It was apparently intended to serve as an introduction for beginners. As we can see from the program of Friedrich-Wilhelms-Universität zu Berlin, the course took place every Saturday from 2 to 3 p.m. in winter semester 1840/41, and it was a "publicum," i.e., without fees. Dirichlet had 38 students in this course, and this number was his personal record (Biermann, 1959, pp. 34–39).

Philipp Ludwig Seidel (1821–1896) studied in Berlin from 1840 to 1842. In his Nachlass, which is kept in the archives of Deutsches Museum in München, we can find several lecture notes of courses given by Dirichlet, and among them the manuscript "Elemente der Lehre von den Reihen" with 78 pages in quarto (Dirichlet, 1840/41). The text is not paginated; at least it is organized in paragraphs.

The Main Topics of Dirichlet's Course

According to Seidel's notes, the main topics of Dirichlet's series course were as follows:

- Basic notions: series, convergence (absolute and conditional) to a "sum", divergence, "Cauchy-criterion" in an epsilonic version
- Elementary examples: geometric series and (alternating) harmonic series, convergence of alternating series

- Invariant limit of absolutely converging series under rearrangement, convergence to different limits or divergence of conditionally converging series under rearrangement (this topic was not covered by Cauchy)
- Basic properties of power series
- Binomial, exponential, logarithmic series (in details deviating from Cauchy's exposition)
- Beginnings of trigonometric series (start with complex numbers), but apparently not completed (lack of time?).

As already indicated, these topics include the most significant ones of the central chapter VI of Cauchy's *Cours*. There are two important differences, however. Dirichlet did not discuss questions regarding continuity in his course, as we will see in an example below. Therefore, Dirichlet's attitude toward Cauchy's famous "mistake" remains unclear. Cauchy had stated in his *Cours* (1821, VI.1, théorème 1) that a convergent series of continuous functions is again continuous, an assertion which was obviously wrong in general, as one could see, e.g., from the Fourier series representing a discontinuous step function. Abel's criticism (1826) of this erroneous theorem became very popular and was often repeated. But even after Cauchy's correction of his statement (1853) some doubts remained as to what Cauchy exactly had meant by his original assertion. For a history of Cauchy's mistake and the subsequent development of the concept of uniform convergence see (Viertel, 2014). A second important point about Dirichlet's course is his discussion of the rearrangement of series, a subject which is absent in Cauchy's *Cours*.

Basic Notions: Cauchy versus Dirichlet

A preliminary remark: Neither Cauchy nor Dirichlet had a concept of real number that was elaborated in a way comparable to later accounts, as, e.g., by Weierstraß or Dedekind. Cauchy at least gave an outline of "nombre," "quantité réelle," and "valeur numérique" in the introduction ("préliminaire") of his *Cours*. Numbers are the result of measurements (which he did not specify more closely), and they are always positive. Real quantities correspond to an increase (then they are positive) or decrease (then they are negative) or an unchanged state (quantity 0) of a measure. The numerical value is equal to the now common absolute value. In Seidel's notes we can find – without further explanation – the terms "Größe" or "Werth" for real quantities, and "Zahlenwerth" or even "absoluter Zahlenwerth" for absolute values. Without use of the absolute sign (which only later became popular, in particular through Weierstraß), Cauchy's as well as Dirichlet's (or Seidel's) exposition is often clumsy or even unclear, and we have to infer from the context whether the absolute value is meant or not.

Convergence and Divergence

In explaining convergence of series, Cauchy resorted to his notion of limit, see (Cauchy, 1821, p. 13):

> Lorsque les valeurs succesivement attribuées à une même variable s'approchent indéfiniment d'une valeur fixe, de manière à finir par en différer aussi peu que l'on voudra, cette dernière est appellé la limite de toutes les autres.

In particular cases Cauchy even considered multivalued limits, derived from the respective limits of subsequences. For example (in modern notation): $\lim_{x \to 0} \sin(1/x) = [-1; 1]$ (Cauchy 1821, p. 14). But in the case of convergent series he (1821, p. 123) demanded a unique limit:

> Si, pour des valeurs de *n* toujours croissantes, la somme s_n[the sum of the first *n* terms of the series] s'approché indéfiniment d'une certaine limite *s*, la série sera dite convergente.

Dirichlet (according to § 2 in Seidel's notes) was clearer in his definition (in English translation):

> [A series is convergent] if the sum of the first *n* terms for an indefinitely increasing *n* approaches a certain quantity, which is called "sum of the series," in such a way that its difference from this limit, if *n* is large enough, becomes smaller than each quantity which is as small as one may assume.

Dirichlet adopted Cauchy's notion of "divergence" in any case of non-convergence. Cauchy (1821, pp. 142–145) discussed the difference between absolute and conditional convergence (the terminology may be due to Weierstraß), but he did not use any specific designations, whereas Dirichlet (§ 7 of Seidel's notes) spoke of "Reihen erster Art" and "Reihen zweiter Art" (series of the first/second kind).

The Cauchy Criterion

The famous "Cauchy-Criterion" – which, by the way, can be found already in writings of Euler and Bolzano (Laugwitz, 1999, p. 187) – providing a necessary and sufficient condition for convergence is stated in the *Cours* on p. 125. A series is convergent if and only if

> … pour des valeurs infiniment grandes du nombre *n*, les sommes
>
> s_n, s_{n+1}, s_{n+2}, &*c*....
>
> diffèrent de la limite *s*, *et par conséquent entre elles*, de quantités infiniment petites.

The phrasing of this condition – the essential part is emphasized, contrary to the original – is not entirely clear. Besides the problem of the somewhat obscure infinitely small quantities (which will be discussed below), the question arises, how far "&*c*.…" should be going. Up to a partial sum s_m with fixed or with arbitrarily large *m*? Only in the following text of the *Cours* this question is clarified by additional explanations.

Dirichlet, on the other hand, provides a stricter formulation (reproduced again according to § 2 of Seidel's notes in English translation). If ρ designates an arbitrarily small fixed positive quantity, then:

> ... one will be able to proceed so far that all following terms, summed up in an arbitrarily great number, cannot yield a result which reaches ρ.

Again, the lack of an appropriate expression for absolute values is obvious, however.

Cauchy made the façon de parler of infinitely small and large quantities explicit in an epsilonic sense in the *préliminaires* to his *Cours*, but in the main text he usually did not refer to it. Dirichlet, on the contrary, avoided such formulations in his series course. According to one place (§ 6) of Seidel's text, he did point out the possibility of using such expressions, however. From his explanations we can see that Dirichlet was far from interpreting Cauchy's pertinent statements in the strict sense of infinitely small or large quantities, an interpretation which had a considerable impact on the modern historiography of analysis since the 1970s. (Laugwitz, 1987) is a typical and still inspiring example; Viertel (2014, chapt. 4.3) provides a critical analysis focusing on discussions of Cauchy's "mistake".

Both Cauchy and Dirichlet gave the criterion the status of an assertion which is easy to see. They did not discuss that the proof that the criterion is sufficient for convergence is based on a property of real numbers which has to be postulated (or derived from a construction principle).

Rearrangement of Series

This topic of Dirichlet's course is in some sense a follow up to brief remarks by Cauchy (1823, *Résumes analytiques*) and Dirichlet himself (1837, paper on number theory). Cauchy (1823, pp. 57 f.) had shown that the alternating harmonic series, which converges to log (2), could be rearranged to a divergent one. Dirichlet (1837, p. 318) gave an example in which the alternating harmonic series is rearranged with the consequence that it converges to a limit different from log (2). He also noted that only in the case of absolutely convergent series the sum would be invariant with respect to any sort of rearrangement.

In his course (§ 7), Dirichlet discussed the similar problem

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\cdots$$

He showed by elementary estimations that the sum of the usual alternating harmonic series minus the sum of this series is greater than $1/4$. Moreover, he also provided a proof of the rearrangement theorem for absolutely convergent series (§ 5 of Seidel's notes). Pringsheim (1898, pp. 91–94) in his survey of conditional versus absolute convergence attributed this proof to Wilhelm Scheibner (1860, pp.

10–12), who in fact treated the more general case of series of complex numbers. Scheibner (1826–1908) studied in Berlin from summer semester 1845 to winter semester 1845/46 and attended some courses given by Dirichlet such as, e.g., those on definite integrals and partial differential equations (according to his curriculum vitae kept in the archives of the University of Halle, UAHW, Rep. 21, Nr. 491). Scheibner later became a professor in Leipzig, where he contributed to a fundamental reform of the mathematics program (Girlich & Schlote, 2009).

Dirichlet: No Use of Continuity in the Series Course

We recall Cauchy's erroneous assertion: Every convergent series of continuous functions is continuous. This would correspond to an interchange of a limit with respect to a variable and an infinite summation in any case.

Cauchy uses this principle of interchange of two limit processes in the *Cours* in many cases. A typical example is the derivation of the exponential series. Cauchy (1821, pp. 164–169) bases his arguments on the binomial series
($\mu \in \mathbb{R}$ and $x \in (-1; 1)$):

$$(1+x)^{\mu} = 1 + \frac{\mu}{1}x + \frac{\mu(\mu-1)}{1\cdot 2}x^2 + \cdots + \frac{\mu(\mu-1)(\mu-2)\cdots(\mu-n+1)}{1\cdot 2\cdots n}x^n + \cdots$$

By means of the substitutions $x \leftarrow \alpha x$ and $\mu \leftarrow 1/\alpha$ he obtains

$$(1+\alpha x)^{1/\alpha} = 1 + \frac{x}{1} + \frac{x^2}{(1\cdot 2)(1-\alpha)} + \frac{x^3}{(1\cdot 2\cdot 3)(1-\alpha)(1-2\alpha)} + \cdots$$

for $x \in (-1/\alpha\,;\ 1/\alpha)$. The limit for $\alpha \to 0$ on both sides yields

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots$$

for $x \in (-\infty; +\infty)$. On the right side we have an unjustified interchange of infinite summation and the limit $\alpha \to 0$.

Dirichlet, on the other hand, starts with the series

$$\varphi(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!},$$

and by means of the quotient criterion he shows that the series is convergent (§ 10 of Seidel's notes). By elementary considerations, which involve, however, rearrangement, he derives the relation

$$\varphi(x)\varphi(y) = \varphi(x+y).$$

From this by induction

$$\varphi(m) = \varphi\,(1)^m$$

follows for natural *m*. Finally, it is shown that

$$\varphi(x) = \varphi\,(1)^x$$

even for all rational *x*. Dirichlet did not discuss the transition to irrational *x*, a process which would have required a special justification of the limits applied. The constant *e* is introduced by the series expansion for $x = 1$.

Cauchy and Dirichlet followed familiar paths. What is interesting, however, is their different approach.

Another Adaption of Cauchy's *Cours*: Schlömilch's *Handbuch*

In order to more precisely discern Dirichlet's particular didactic approach, it is opportune to place his account in the broader context of other attempts to render core subjects of Cauchy's *Cours* more elementary. In German language, one work became especially prominent: Schlömilch's *Handbuch der mathematischen Analysis*, erster Theil: algebraische Analysis, whose first edition appeared in 1845, and which experienced a total of 6 editions by 1881. The title of this book followed Cauchy's model: "First part" is emphasized, but a second part of the *Handbuch* was never published.

Oskar Schlömilch (1823–1901) was an eminently busy propagator of a new and more rigorous mathematics, where he relied in particular on Cauchy, and a very well-respected organizer of teaching mathematics. He taught first at the Polytechnic School in Dresden, and then became a senior official at the Saxonian Ministry of Education. During his studies he had spent two semesters from winter semester 1840/41 to summer semester 1841 at the University of Berlin, where he also attended two courses held by Dirichlet (partial differential equations and number theory, see Abgangszeugnis Nr. 219, Humboldt Universität, Archiv).

Unlike Dirichlet, Schlömilch treats Cauchy's algebraic analysis as a whole. Roughly, he covers the same subjects as Cauchy, and toward the end of the *Handbuch* one can also find a rather comprehensive account of continued fractions which are not included by Cauchy. Schlömilch's own didactic goals can be clearly seen from the two following quotations (translated into English) out of the preface of the first edition:

> We find in Cauchy's work [...] the maximum rigor, though combined with much brevity in the exposition; nevertheless, the beauty of architecture suffers from a very artificial organization, and the life of invention is entirely absent ….
>
> I have [...] tried to take a middle course, which can reconcile the interest of the heuristic line of thought with the rigor of the French analyst.

Moritz Stern (1807–1894), professor in Göttingen, wrote as early as 1845 (translated into English):

> We agree with the author [= Schlömilch] in his criticism, but we are not able to confirm that he succeeded in the desired conciliation. On the contrary, we have observed so little of it that we would not have recognized his effort if he had not explicitly expressed his intentions (Stern 1845, p. 3).

Around 1900, in obituaries appreciating Schlömilch's life and work, he was praised as a promulgator of mathematical rigor as it was rarely usual in teaching around 1850, but, on the other hand, it was emphasized that his textbooks, most notably the *Handbuch*, had their success mainly in the training of prospective engineers at polytechnic schools, see, e.g., (Cantor, 1901; Helm, 1901). From the point of view of an historian, Jahnke (1990, p. 319) characterizes Schlömilch's *Handbuch* by the following words (quoted in English translation):

> Cauchy's theory was re-shaped into the *elementary*, and it was reduced to the aspect of a method of safe calculation.

And indeed, a closer inspection of the first and second editions of the *Handbuch* yields the impression that "elementary approach" as well as "safe calculation" are provided by a bunch of examples and counterexamples rather than by a clearly visible theory.

Basic notions such as "limit, function, continuity" are explained by many long-winded examples, but less by unambiguous and complete definitions. Schlömilch (1845, § 5) discusses the notion of "limit" for "functions" that do or don't "approach" a certain quantity if the independent variable is "continually" increasing to infinity or decreasing to 0. The following quotations (translated into English) from his explanations of "limit" (1845, pp. 21 f.) may give the reader an impression of Schlömilch's particular mode of exposition:

> One can distinguish at least three cases [...] In order to have examples for these three cases, let us consider the three functions
>
> $$a + x, \sin x\, , a + \frac{1}{x}$$
>
> for continually increasing x.
>
> [...]
>
> Finally, in the third function for increasing x the quotient $1/x$ always decreases. For, if we set in this order $x = n, n + 1, n + 2, \ldots$, where n is any positive number, then we have
>
> $$\frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+2} > \frac{1}{n+3} \cdots$$
>
> But this decreasing goes so far that $1/n$ becomes smaller than any quantity which can be given, no matter how small it is. Namely, if we want to have $1/n < \rho$, where ρ means any small quantity, we only have to set $n > 1/\rho$ in order to immediately satisfy the requirement. The limit which $1/n$ approaches must therefore be zero. Thus, the function $a + 1/x$ for increasing x continually approaches the limit a.

Indeed, Schlömilch's change from real to natural arguments and back, which was apparently due to his quest for didactic simplifications, in the end rather disguised a clear and "rigorous" understanding of the limit in question.

Based on the concept of limit, convergence and divergence of series are explained in the usual manner in Schlömilch's book (1845, § 20); in § 19-20 there are again various introductory examples, and very interesting ones among them, but a general definition for the "sum" of a convergent series is lacking, and the Cauchy criterion cannot be found.

Concerning the power series of the exponential function, Schlömilch even gave three different derivations, one (1845, § 32) apparently found by himself, and the other two (1845, § 33) as outlined above. As far as analytical rigor is concerned, Schlömilch confined himself in all these derivations to the assessment that the applied series are convergent in every form and after every transformation. He did not bother about rearrangement or processes with parameters tending to a certain limit. On the one hand, this way of reasoning reflects Schlömilch's main concern to base all considerations on convergent expansions, combined with a strict rejection of the use of divergent ones. This point was widely discussed. On the other hand, the implicit message of this practice was: You can do anything with a convergent series. But the message was wrong. At least in the first editions of the Handbook, continuity arguments, such as those needed for the transition from rational to irrational arguments, were notoriously incomplete.

With regard to the variety of its contents, the book was overloaded. As a result, basic notions and fundamental theorems remained rather in the background. Jahnke (1990, pp. 310–320) explains these and similar deficits in Schlömilch's accounts by the need for compromises between the common praxis of teaching and doing mathematics on the one hand, and Cauchy's approach, which was often considered too complex and radical, on the other. Schlömilch's didactic approach was to gently introduce the reader to the theory through examples of increasing difficulty and complexity. In this sense, the theory was rather a collection of problems whose solutions could be applied in an analogous way to similar problems. This was the inductive approach, which in contemporary terms and with a certain right might have been called "elementary." At the same time, Schlömilch wanted to introduce an essential aspect of analytical rigor and thus of a "safe" computational method by his prohibition to use divergent series expansions. In this way, he undoubtedly promoted his reader's ability to solve problems and apply mathematical methods. This may be the reason why his work, which could serve as both a textbook and a companion book, came to be so highly regarded by engineers in later times.

Dirichlet's Elementary Approach

Notwithstanding all the differences, Schlömilch's book and Dirichlet's course both pursued the same goal of an "elementary" didactic elaboration of Cauchy's analytic theory. Here, "elementary" can be understood in the literal sense, that is, as an approach that attempts to build up the theory from individual small building blocks. The elements to which Schlömilch referred were primarily examples and counterexamples. This was a very popular way of teaching. Dirichlet, on the other hand, chose a completely different approach in his lectures in order to make things elementary: a smart selection of subjects, as the exposition was restricted to a small set of basic notions and definitions – without continuity – which constituted the building blocks. In this way, the theoretical elements are clearly visible (absolute versus conditional convergence, Cauchy criterion, rearrangement, etc.) Examples are given – sometimes even beyond the scope of Cauchy, as with respect to calculating logarithms or higher roots –, but they serve as an illustration and motivation of the theory and not as a substitute for it. We have a strict emphasis on absolutely necessary and "safe" theoretical "elements," in accordance with the restriction to rigorously founded topics to which Cauchy alludes in the preface of the *Cours*. But Dirichlet developed an improved, more direct and less intricate exposition compared to Cauchy's.

The development of real analysis from ca. 1820 to ca. 1880 was not a linear process starting with Cauchy and ending with Weierstraß. Different people with different opinions and goals were involved. Nevertheless, the development of Cauchy's ideas played a decisive role in this process. In this sense, Schlömilch in his textbooks certainly also contributed to this development, at least through his emphasis on being cautious about formal and therefore possibly divergent series expansions. However, the approach taken by Dirichlet in his lectures seems to have been more productive. The course on series is an especially well-fitting example of the combination of rigor and elementary exposition following Cauchy's model. In other courses with more advanced topics, this ideal could only be realized by Dirichlet in an exemplary way. In the course on integrals, see (Arendt, 1904), a very careful discussion of the one-dimensional integral was followed by a rather intuitive introduction into integrals in two or three dimensions. Whereas Schlömilch's books reached a large audience, but ultimately served other purposes than originally intended, Dirichlet's courses addressed a relatively small number of students only. However, a considerable number of these students later on became very influential in the development of mathematics, and especially of real and complex analysis. As we have seen in particular by the course on series, Dirichlet set good examples, or, as Lorey (1916, p. 71) put it, he created "classic patterns" for the teaching of mathematics.

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